ARISTOTLE UNIVERSITY OF THESSALONIKI

DEPARTMENT OF CIVIL ENGINEERING

LABORATORY OF STATICS & DYNAMICS

LECTURE NOTES

ON

THE BOUNDARY ELEMENT METHOD (BEM) IN ENGINEERING MECHANICS & ELASTODYNAMICS

BY

GEORGE D. MANOLIS, PROFESSOR

PRESENTATION TO THE DEPARTMENT OF ASTRONOMY, PHYSICS OF THE EARTH AND METEOROLOGY

FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

COMENIUS UNIVERSITY OF BRATISLAVA, SLOVAKIA

6 FEBRUARY 2008

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SUMMARY

For complex problems in engineering, analytical solutions are difficult and often impossible to obtain. Use must then be made of numerical methods of solution for the determination of approximate solutions that can be used in engineering practice. It must be kept in mind, however, that even the mathematical idealization of an engineering system is in itself an approximate representation of reality.

A numerical method is either of the differential or integral type, depending on whether the analysis precedes or follows integration of the governing differential equation of the problem at hand. The most widely known and used method at present in structural mechanics is the Finite Element Method (FEM), with the Finite Difference Method (FDM) being popular in fluid mechanics. The latter method replaces the governing differential equation by a system of algebraic equations valid at a set of nodes (or nodal points) within the domain of interest, through an approximation of the derivatives by finite differences. The former method replaces the domain itself by a set of sub-domains (known as elements) and then approximates the behavior of the field variable(s) within each element. The elements are interconnected at the nodes and comprise the entire domain.

The Boundary Element Method (BEM) is in essence a numerical integration of integral equation representations of the governing differential equations. There is an intermediate step, however, whereby the integral equation is reformulated so that actual boundary-value problems can be solved, a process that results in what is known as Boundary Integral Equation (BIE) formulations. In many cases, the BEM requires a surface-only discretization of the problem at hand, in contrast to both FEM and FDM that require full domain discretizations. Finally, from a chronological point of view, the FDM predates all, with FEM and BEM developments dating from the 1960's and 1980's, respectively.

Of course, hybrid numerical schemes are possible through combinations of the three basic classes of methods previously mentioned, not to mention other possibilities such as the finite strip method, meshless methods, etc. The question of which numerical approach is best for a given problem is, in many ways, an open question to be answered by engineers on the basis of professional experience.

In this brief, introductory-type treatise, we will present the basic concepts underlying the BEM and show how it can be applied in two important fields, namely potential problems and elastostatics. Of course, in this last quarter-century, the BEM has been used across the entire spectrum of engineering science (e.g., structural mechanics, fluid mechanics, wave propagation, electromagnetic theory, acoustics, fracture mechanics, corrosion, etc.), as can be ascertained by perusing through books on the BEM that have appeared in recent years.

1. INTRODUCTION

1.1 Brief Review of the Literature

The theory behind integral equations was developed by Fredholm (1903), while applications followed later in potential theory (Kellogg, 1929) and elastostatics (Muskhelishvili, 1953; Kupradze, 1965). A comprehensive treatise on the theory of singular integral equations is that of Mikhlin (1965), while the numerical treatment of integral equations was elaborated later on in books such those by Baker (1977) and Delves and Mohamed (1992).

The earliest numerical methods for the solution of engineering problems based on integral equations are attributed to Trefftz (1926) and to Prager (1928). It was only after the introduction of the electronic computer as a computational tool in the mid-1950's that a new impetus was given to the subject of computational methods. Specifically, we have the emergence of boundary integral methods in potential theory (Smith and Pierce, 1958; Hess, 1962; Jawson, 1963; Symm, 1963), in acoustics (Friedman and Shaw, 1962; Banaugh and Goldsmith, 1963), and in elastostatics (Massonnet, 1965; Oliveira, 1968). At about the same time, boundary integral methods of the direct type, which were extremely well suited for the solution of boundary value problems (BVP), were introduced in two-dimensional (2D) and three-dimensional (3D) elastostatics by Rizzo (1967) and Cruse (1969), respectively, and in transient elastodynamics by Cruse and Rizzo (1968). Also, the relation between integral methods and the general weighed residual statement was identified by Zienkiewicz et al. (1977) and by Brebbia (1980).

Numerous articles have since appeared on the use of numerical methods based on boundary integral equation formulations in all fields of engineering science such as potential theory, potential fluid flow, acoustics, torsion, electric and magnetic field theories, elastostatics, elastodynamics, plates and shells, heat conduction, viscoelasticity, thermoelasticity, fracture mechanics, plasticity, water waves, viscous flow, ground water flow, corrosion, contact problems, coupled-field problems, etc., as can be surmised by looking through a number of textbooks that have appeared in the last twenty years or so (Jawson and Symm, 1977; Liggett and Liu, 1983; Brebbia et al., 1984; Beskos, 1987; Manolis and Beskos, 1988; Brebbia and Dominguez, 1989; Beskos, 1991; Antes and Panayiotopoulos, 1992; Pozrikidis, 1992; Dominguez, 1993; Kane, 1994; Banerjee, 1994; Melnikov, 1995; Canas and Paris, 1997; Sladek and Sladek, 1998; Bonnet, 1999).

1.2 Development of the BEM

Using BEM, a problem is formulated in terms of integral equations relating boundary values and solution is then found numerically. If values in the interior domain are needed, they can be calculated from the boundary data. Thus, the dimension of the problem at hand is reduced by one, while the resulting system of algebraic equations is invariably smaller in size compared to similar systems obtained from use of the FEM or the FDM. Only in certain cases (e.g., nonlinearities, inhomogeneities, certain types of body forces) does the BEM require discretization of the interior domain. There are two basic categories of BEM, namely indirect and direct. In the former case, which is also known as the source method, the boundary integral formulation is accomplished in terms of fictitious source densities from which the physical quantities of the problem can be recovered. In the latter case, the boundary integral formulation is accomplished in terms of physical quantities such as potentials, fluxes, displacements, etc., which are computed directly without use of intermediate steps.

Fredholm (1903) established the existence of integral equation solutions to potential problems on the basis of a limiting discretization procedure and identified the Fredholm integral equations (FIE) of the first, second and third type. The first application of a direct BEM was in fluid mechanics (and specifically for the axisymmetric jet problem) by Trefftz (1926), who used the method of successive approximations to satisfy his integral equation. Prager (1928) examined doubly symmetric potential flow past an elliptic cylinder using a direct-type boundary integral formulation, and subsequently divided the surface of the problem into elements, thus reducing the integral equations into a system of algebraic equations.

It appears that boundary integral techniques were well known by the 1950's, but were not popular due to high computational effort. Among the first to use the electronic computer to solve axisymmetric potential fluid flow problems were Smith and Pierce (1958). Martensen (1959) extended Prager's method to 2D fluid flow problems, while Hess and Smith (1964) solved 3D potential fluid flow problems using an indirect BEM called the panel method because the surface of the body in question was discretized into flat quadrilateral elements. At about the same time, Jawson (1963) and Symm (1963) presented a numerical technique for solving Fredholm integral equations of the first kind resulting from use of the indirect BEM in 2D potential problems. They also presented a direct BEM based on Green's third identity, which was capable of relating in integral form the physical quantities of the general mixed BVP.

Since the mid-1960's, a number of both direct and indirect BEM formulations for elastostatic problems have appeared. The most notable of these is the direct formulation of Rizzo (1967) relating boundary displacements and tractions, which allowed for the solution of general, mixed-BVP. The early 1970's witnessed an extension of the BEM to other engineering fields, the introduction of higher–order elements which brought about significantly improved accuracy and efficiency, as well as the coupling of the BEM with other methods such as the FEM in an effort to exploit the best features of different numerical techniques. Nowadays, the BEM and has become an established computational tool whose efficiency for certain categories of problems is well known and complements the most popular method to date, namely the FEM.

1.3 Mathematical Background

Green's second theorem relates two C^2–continuous functions u and v, defined over a finite and simply connected domain V with a regular surface S as

$$\int_V \left(u\nabla^2 v - v\nabla^2 u\right) dV = \int_S \left(u\frac{dv}{dn} - v\frac{du}{dn} \right) dS \qquad (1.1)$$

Although the above equation is nothing more than a reciprocal relationship between u and v, it nevertheless forms the basis of the direct BEM.

Fredholm's integral equations (FIE) of the first and second kind are respectively defined as

$$\int_S u(\xi)K(x,\xi)dS(\xi) = v(x) \qquad (1.2)$$

and

$$u(x) - \int_S u(\xi)K(x,\xi)dS(\xi) = v(x) \qquad (1.3)$$

with K a known kernel function and u, v being the unknown and known functions.

The Fredholm alternatives are as follows:

Theorem 1: Either Eqn. (1.3) has one, and only one, solution for every function *v*, or the corresponding homogeneous equation *(v = 0)* has at least one non-trivial solution $(u \neq 0)$.

Theorem 2: If the first alternative holds, then the next alternative also holds for the adjoint equation

$$\bar{u}(x) - \int_S \bar{u}(\xi) K(x,\xi) dS(\xi) = v(x) \qquad (1.4)$$

For either alternative, the corresponding homogeneous equation and its adjoint have the same finite number of linearly independent solutions.

Theorem 3: In case of the second alternative, a necessary and sufficient condition for the existence of a solution for Eqn. (1.3) is that

$$\int_S v(\xi) \bar{u}(\xi) dS(\xi) = 0 \qquad (1.5)$$

where $\bar{u}(\xi)$ is any solution of Eqn. (1.4). The above three theorems can be used to prove existence and uniqueness of solution for the classical Dirichlet and Neumann BVP of potential theory.

Despite the fact that Eqn. (1.2) looks simpler than Eqn. (1.3), it was only recently that its proper theoretical background was established. In 2D, Muskhelishvili (1953) proved the following theorem:

Theorem: Either equation

$$\int_S u(\xi)\ln|x-\xi|\,dS(\xi)=1 \qquad (1.6)$$

or equation

$$\int_S u(\xi)\ln|x-\xi|\,dS(\xi)=0 \qquad (1.7)$$

has a solution for any smooth contour S. This theorem can then be used to establish existence and uniqueness of solutions for Eqn. (1.2) in 2D.

The kernel function $K(x, \xi)$ is singular if

$$\lim \quad r^a(x,\xi)K(x,\xi) \to 0 \quad as\ \xi \to x \tag{1.8}$$

where r is the radial distance between points ξ and x, while a is a real number. As example, a kernel in 3D potential theory is $K_1 = u^* = r^{-1}$ and its derivative is $K_2 = du^*/dn$, i.e.,

$$K_2 = \frac{du^*}{dn} = \frac{du^*}{dr}\frac{dr}{dn} = \frac{d\left(r^{-1}\right)}{dr}\frac{dr}{dn} = -\frac{1}{r^2}\frac{dr}{dn} \tag{1.9}$$

where $\underline{n}$ is the outward pointing normal vector at ξ. For K_2 and $a = 2$, Eqn. (1.8) reduces to

$$\lim_{\xi \to x} \quad \frac{dr(x,\xi)}{dn(\xi)} \to 0 \tag{1.10}$$

Thus, kernels u^* and du^*/dn for 3D potential problems are singular, which means the Fredholm theorems apply.

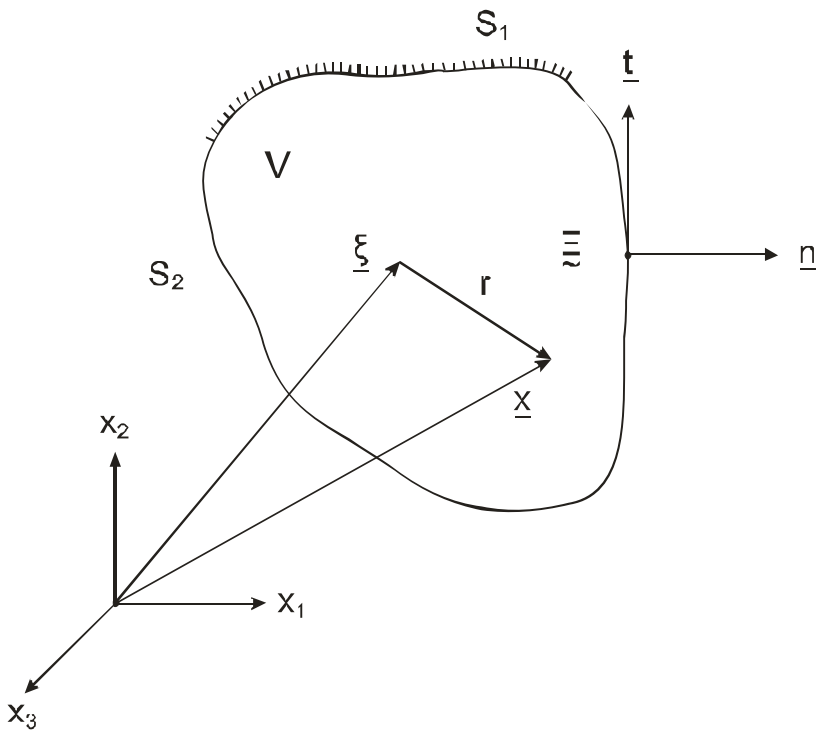

Figure 1: Geometry of a continuous body with volume *V* and surface *S*

1.4 Advantages / Disadvantages of the BEM

(a) It is a numerical technique founded on a firm mathematical basis and is therefore applicable to a wide range of problems in engineering.

(b) It usually requires surface discretization only, which reduces the dimensionality of the problem at hand by one.

(c) The method is capable of yielding a high degree of accuracy, since the underlying integral equation is an exact statement of the problem at hand. Approximations are introduced at the second stage, i.e., when the surface of the problem is discretized.

(d) The fundamental singular solutions (or Green's functions) used in the BEM satisfy radiation-type boundary conditions. As a result, infinite or semi-infinite domains can be treated without the artificial boundaries introduced in the FEM and the FDM.

(e) Values of the unknowns at points inside the domain can be selectively found from the boundary solution.

(f) The BEM can be used to construct influence matrices for large homogeneous regions or regions containing semi-infinite boundaries (super elements), which in turn can be incorporated within FEM analyses.

(a) The BEM generates non-symmetric, fully populated influence matrices. This is in contrast to the FEM, which generates symmetric and sparse influence matrices. The order of the BEM influence matrices, however much smaller than that of those generated by the FEM.

(b) For certain classes of problems, it is either very difficult or impossible to obtain the appropriate fundamental solutions. It is also possible that the fundamental solution is an approximate one, or is known in discrete form only, thus giving rise to rather inefficient computational schemes.

(c) The BEM is inefficient as compared to the FEM for problems where one spatial direction is small compared to the remaining ones, and for problems with rapidly varying material properties.

(d) The presence of known distributed body forces in linear problems or pseudo-incremental body forces in nonlinear problems requires volume integrals, which in turn necessitate internal domain discretization.

2. INTEGRAL EQUATION FORMULATIONS

Consider the non-homogeneous partial differential equation (PDE)

$$L\{u(\underline{x})\} = g(\underline{x}) \qquad (2.1)$$

in volume V, where $u(\underline{x})$ is the dependent variable and $g(\underline{x})$ is the forcing function. We first consider the homogeneous *(g = 0)* version of the PDE and define the inner product

$$\langle Lu, w\rangle = \iiint L\{u(\underline{x})\}w(\underline{x})\,dV \qquad (2.2)$$

where $w(\underline{x})$ is an arbitrary function to be chosen. Successive integrations by parts yields the following statement:

$$\iiint L(u)w dV = \iint \{N^*(w)M(u) - M^*(w)N(u)\}\,dS + \iiint uL^*(w)\,dV \qquad (2.3)$$

In the above, N and M are differential operators resulting from the integrations and the asterisc superscripts denotes the adjoint operator. If a differential operator has the property that $L^* = L,$ then it is said to be self-adjoint. Self-adjointness in an operator is analogous to symmetry in a matrix, and results in $N^* = N$, $M^* = M$. Also, the above integration process results in two types of boundary conditions, essential (or kinematic) and non-essential (or natural), which are respectively defined as *M(u)* on the S_1 part of the surface and *N(u)* on the remaining S_2 part. The presence of the forcing function in the PDE results in a term $\iiint gL^*(w)\,dV$ on the right hand side (RHS) of Eqn. (2.3) that remains unaltered, unless $g(\underline{x})$ has a special form. The development that follows is limited to self-adjoint, positive definite operators (i.e., $<Lu, u> > 0$ if $u \neq 0$ and $= 0$ if $u = 0$).

We will focus here on the Laplacian ∇^2, which is a self-adjoint, positive definite operator. The form that equivalent to Eqn. (2.3) is nothing more than Green's identity, i.e.,

$$\iiint \left(u\nabla^2 w - w\nabla^2 u\right) dV = \iint \left(\frac{\partial u}{\partial n} w - u\frac{\partial w}{\partial n}\right) dS \qquad (2.4)$$

The above statement is not as useful as it first seems for the construction of integral equation statements, because of the presence of the volume term $\iiint u\nabla^2 w dV$. The level of effort needed to dispose of this term is comparable to solving the original equation, i.e., to set $\nabla^2 w = 0$ and then find *w*. As a result, recourse is made to the method of weighted residuals (MWR). In its most general form, the MWR for Laplace's equation with a non-zero forcing function is (Brebbia et al., 1984)

$$R(\tilde{u}) = \iiint \left(\nabla^2 \tilde{u} - g\right) w dV + \iint (\tilde{u} - \bar{u}) \frac{\partial w}{\partial n} dS_1 + \iint (\tilde{q} - \bar{q})\ w\ dS_2 \to 0 \qquad (2.5)$$

as $\tilde{u} \to u$. The above equation implies that three types of errors are minimized as we seek the solution $u(\underline{x})$, namely the error committed in (a) satisfying the differential operator $\nabla^2 u - g = 0$, (b) satisfying the kinematic boundary conditions $\bar{u}$ on S_1, and (c) satisfying the natural boundary conditions $\bar{q}$ on S_2. Although the above formulation is meant for 3D problems, it is possible to construct a 2D counterpart, with the volume and surface integrals respectively replaced by surface and line integrals.

A number of approximate techniques result from the general MWR statement of Eqn. (2.5), depending on the choice of the weighting function w and on the boundary conditions that $\tilde{u}$ and w have to satisfy. The variant of the MWR where $\tilde{u}$ and w are the same function is called Galerkin's method (or, more appropriately, the MWR with Galerkin's criterion). It is very commonly used in deriving FEM formulations, and exclusively so in the absence of a functional whose Lagrange-Euler equation is the original differential equation, in which case a variational technique can be employed (Zienkiewicz, 1977). In retrospect, the novelty behind the FEM was in satisfying Eqn. (2.5) on an element-to-element basis rather than on the entire domain of definition of the problem. This considerably simplifies the choice of representations for $\tilde{u}$ (usually as a summation of known basis function Φ_i with unknown constants a_i to be determined) and tremendously improves the accuracy of the method. The volume integral in Eqn. (2.5) cannot be avoided and actually gives rise to the so-called system matrix. Although the FDM is based the concept of defining a series of nodes at which a discrete version of the differential operator is satisfied in an average sense, it is possible to derive FDM formulations using the MWR over sub-regions with the weighting function expressed in terms of Dirac delta functions (Brebbia et al., 1984).

The power behind the direct BEM approaches is in dispensing with the volume integral of Eqn. (2.5) by identifying the weighting function with a generalized Green's function of the differential operator. Double integration by parts of the term $\iiint\left(\nabla^2\tilde{u}\right)wdV$ results, upon re-arrangement, in the following expression:

$$\iiint\left(\nabla^2 w\right)\tilde{u}dV = \iint\bar{u}\frac{\partial w}{\partial n}dS_1 + \iint\tilde{u}\frac{\partial w}{\partial n}dS_2 - \iint\tilde{q}wdS_1 - \iint\bar{q}wdS_2 + \iiint gwdV \qquad (2.6)$$

By identifying w as the solution of $\nabla^2 w = -4\pi\delta$ and under homogeneous (radiation type) boundary conditions, the volume integral on the left hand side simply becomes $-4\pi\tilde{u}$. In case where $g = 0$, then Eqn. (2.6) involves only boundary terms. If not, then the volume term is unavoidable, unless special forms of g are present, which allow a further reduction into surface integrals. Once Eqn. (2.6) becomes a boundary integral equation following the path previously mentioned, then the unknown boundary quantities $\tilde{u}$ and $\tilde{q}$ can be solved in terms of the boundary conditions $\bar{u}$ and $\bar{q}$. Substitution in the original integral equation statement gives values of the potential at any desired location within the volume.

The Trefftz (1926) method is an early version of the direct BEM whereby $u = w$. Going back to the original Green's identity given in Eqn. (2.4) and imposing the boundary conditions gives

$$\iint q\bar{u}dS_1 + \iint \bar{q}udS_2 = \iint \bar{u}\frac{\partial u}{\partial n}dS_1 + \iint u\frac{\partial \bar{u}}{\partial n}dS_2 \qquad (2.7)$$

This can now be used for solution of the dependent variables of the problem *u, q* in terms of the prescribed boundary conditions. There are finally two additional methods for deriving boundary integral equation-based formulations. The first is the indirect BEM, which is based on physical insight regarding the differential operator at hand. The other is a variant of the direct BIEM technique that is based on a reciprocal relationship (if available) that relates two distinct fields, one of which is the solution sought while the other is identified with the generalized Green's function. An example from elastostatics is Somigliana's identity, which is based on the reciprocal work concept.

3. THE BEM IN POTENTIAL PROBLEMS

3.1. Potential Theory

Consider a C^2 function $p = p(\underline{x})$ satisfying Laplace's equation

$$\nabla^2 p(\underline{x}) = \frac{\partial^2}{\partial x_i \partial x_i} p(\underline{x}) = 0 \qquad (3.1)$$

in the 3D, simply connected domain V of Fig. 1, bounded by a closed smooth surface S with the following boundary conditions (BC):

$$p(\underline{x}) = \overline{p}(\underline{x}) \quad \text{on} \quad S_1 \qquad (3.2)$$

$$\frac{\partial p(\underline{x})}{\partial n} = \nabla p(\underline{x}) \cdot \underline{n} = \overline{f}(\underline{x}) \quad \text{on} \quad S_2 \qquad (3.3)$$

In the above, $\underline{n}$ is the outward pointing, unit normal vector on $S = S_1 U S_2$. Mixed BC are of the Cauchy-type that combine both Dirichlet and Neumann BC, as given by Eqn. (3.2) and Eqn. (3.3), respectively. Function $p(\underline{x})$ is called harmonic if it is continuous in V and S, C^2 differentiable in V and satisfies Laplace's equation.

One special case in potential theory arises for a mass distribution with density $\rho(\underline{x})$ in V, whereby the attractive forces create a volume potential of the type

$$\rho(\underline{x}) = \int_V \rho(\underline{\xi}) \frac{1}{r(\underline{x}, \underline{\xi})} dV(\underline{\xi}) \qquad (3.4)$$

If there is a mass distribution of density $\mu(\underline{x})$ on S, the attractive forces create a surface (single–layer) potential of the type

$$\rho(\underline{x}) = \int_S \mu(\underline{\Xi}) \frac{1}{r(\underline{x}, \underline{\Xi})} dS(\underline{\Xi}) \qquad (3.5)$$

The potential obtained by taking the limit of two single layer potentials of opposite sign, which coincide in the limit, is called a double–layer potential and has the form

$$\rho(\underline{x}) = \int_S \lambda(\underline{\Xi}) \frac{d}{dn(\underline{\Xi})} \frac{1}{r(\underline{x}, \underline{\Xi})} dS(\underline{\Xi}) \qquad (3.6)$$

where $\lambda(\underline{\Xi})$ is the new surface density.

Potential functions (3.4)-(3.6) are defined in a 3D space. In 2D, the *ln(r)* replaces the *1/r* function, while *V* and *S* are replaced by surface *S* and contour *C*, respectively. It can be shown that any harmonic function can be expressed as a potential distribution and vice-versa.

Equations (3.5) and (3.6) contain singularities in their integrand as $r(\underline{x},\underline{\Xi})$ approaches zero, i.e., as $\underline{x} \to \underline{\mathrm{X}} \to \underline{\Xi}$ from inside *V* in the direction of $\underline{n}$ (see Fig. 2). Consider the former case first; through a limiting procedure we have that

$$p(\underline{x}) = \lim_{\varepsilon \to 0} \left\{ \int_{S-S_e} \mu(\underline{\Xi}) \frac{1}{r(\underline{x},\underline{\Xi})} dS(\underline{\Xi}) + \int_{S_e} \mu(\underline{\Xi}) \frac{1}{r(\underline{x},\underline{\Xi})} dS(\underline{\Xi}) \right\} \qquad (3.7)$$

It is clear that the integral over $S - S_\varepsilon$ is continuous as $\varepsilon \to 0$. The integral over S_ε contains a weak singularity and is also continuous as $\varepsilon \to 0$, provided that density μ remains bounded at all points along *S*. Next, consider Eqn. (3.6), where again through the same limiting procedure we obtain

$$p(\underline{x}) = \lim_{\varepsilon \to 0} \left\{ \int_{S-S_e} \lambda(\underline{\Xi}) \frac{d}{dn(\underline{\Xi})} \frac{1}{r(\underline{x},\underline{\Xi})} dS(\underline{\Xi}) + \int_{S_e} \lambda(\underline{\Xi}) \frac{d}{dn(\underline{\Xi})} \frac{1}{r(\underline{x},\underline{\Xi})} dS(\underline{\Xi}) \right\} \qquad (3.8)$$

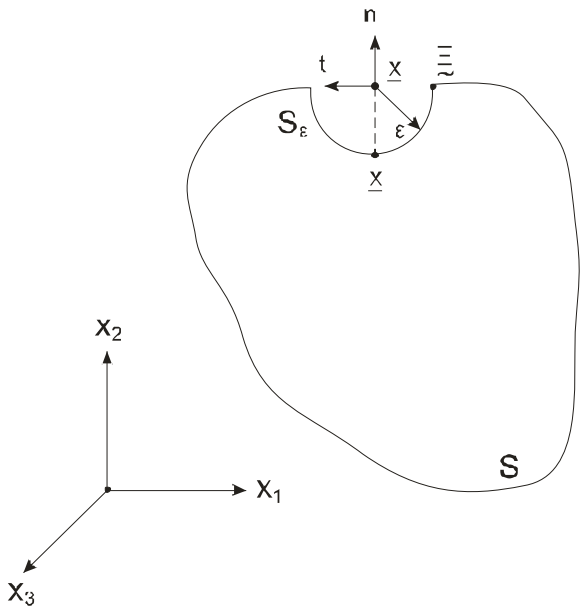

Figure 2: Source-receiver configuration as field point *x* approaches the surface *S*

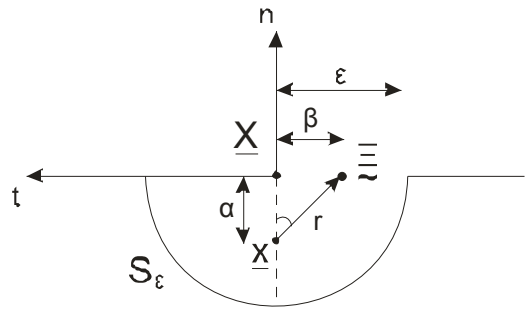

Figure 3: Details of the volume of exclusion on the surface for singularity integration as source and receiver points coalesce

Once more the integral over $S - S_\varepsilon$ remains continuous as $\varepsilon \to 0$, since $r \neq 0$. The integral over S_ε, however, contains a strong singularity that can be written as

$$\int_{S_\varepsilon} \{\lambda(\Xi) - \lambda(\underline{x})\} \frac{d}{dn(\Xi)}\left(\frac{1}{r(\underline{x},\Xi)}\right) dS(\Xi) + \lambda(\underline{x}) \int_{S_\varepsilon} \frac{d}{dn(\Xi)}\left(\frac{1}{r(\underline{x},\Xi)}\right) dS(\Xi) \qquad (3.9)$$

If density λ is continuous on S, then the first integral in Eqn. (3.9) vanishes as $\varepsilon \to 0$. We now look at S_ε in greater detail, as shown in Fig.3. For the second integral, we have that

$$\lambda \int_{S_\varepsilon} \frac{d}{dn}\left(\frac{1}{r}\right) = \lambda \int_{S_\varepsilon} \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{dn} dS = \lambda \int_{S_\varepsilon} \left(-\frac{1}{r^2}\right)\left(\frac{a}{r}\right) dS =$$

$$-\lambda \int_0^{2\pi}\int_0^{\beta} \frac{a}{r^3} \beta d\beta d\theta = -2\pi\lambda \int_a^{\sqrt{\varepsilon^2+a^2}} \frac{a}{r^3} r dr = -2\pi\lambda \int_a^{\varepsilon} \frac{a}{r^2} dr = 2\pi\lambda \frac{a}{r}\Big|_a^{\varepsilon} = 2\pi\lambda\left(\frac{a}{\varepsilon} - 1\right) \qquad (3.10)$$

In the limit as $\varepsilon \to 0$, $a \to 0$ and the above result is equal to $-2\pi\lambda.$. If $\underline{x}$ approaches $\underline{X}$ from outside, then $dr/dn = -a/r$ and the limiting process gives $+2\pi\lambda.$. Thus, the 3D double layer potential has a discontinuity or jump of

$$p^+(\underline{X}) - p^-(\underline{X}) = -2\pi\lambda(\underline{X}) - (-2\pi\lambda(\underline{X})) = -4\pi\lambda(\underline{X}) \qquad (3.11)$$

These concepts are valid for 2D problems as well, where the kernel function is *ln(r)*. The discontinuity or jump term here is equal to $-2\pi\lambda(\underline{x})$.

In sum, we see that for $\underline{x} \to \underline{X}$, the single-layer and double-layer surface potentials given in eqs (3.7) and (3.8) become

$$p(\underline{X}) = \int_S \mu(\underline{\Xi}) \frac{1}{r(\underline{X}, \underline{\Xi})} dS(\underline{\Xi}) \qquad (3.12)$$

$$p(\underline{X}) = -4\pi\lambda(\underline{X}) + \int_S \lambda(\underline{\Xi}) \frac{d}{dn(\underline{\Xi})} \left(\frac{1}{r(\underline{X}, \underline{\Xi})} \right) dS(\underline{\Xi}) \qquad (3.13)$$

The kernel functions appearing above are the fundamental solutions $u(\underline{x}, \underline{\xi})$ of Laplace's equation for a point force, i.e.,

$$\nabla^2 u(\underline{x}, \underline{\xi}) = -\delta(\underline{x}, \underline{\xi}) \qquad (3.14)$$

with δ being the Dirac (or generalized) delta function with the following properties:

$$\delta(\underline{x}, \underline{\xi}) = 0 \; for \quad x \neq \xi \quad and \qquad \int_V \rho(\underline{\xi}) \delta(\underline{x}, \underline{\xi}) dV(\underline{\xi}) = \rho(\underline{x}) \qquad (3.15)$$

Specifically, the fundamental solutions are $u(\underline{x}, \underline{\xi}) = (1/4\pi) r(\underline{x}, \underline{\xi})$ in 3D, while $u(\underline{x}, \underline{\xi}) = (1/2\pi) \ln(1/r(\underline{x}, \underline{\xi}))$ in 2D.

Applications of potential theory include the following areas:

(a) Gravitational force field, where ρ is the gravitational potential and $\underline{f} = \nabla\rho$ is the gravitational force.

(b) Potential fluid flow, where $\phi = \rho$ is the velocity potential and $\underline{v} = \nabla\phi$ is the velocity for steady-state irrotational flow of an ideal fluid.

(c) Groundwater flow, where p is the piezometric head and $\underline{v} = -k\nabla\rho$ (k the soil permeability) is the fluid velocity.

(d) Heat conduction, where $T = p$ is the temperature and $h = c\nabla T$ (c the thermal conductivity) is the heat flux.

(e) Electric current flow, where U = p is the electrical potential and $\underline{\underline{i}} = -\sigma\nabla u$ (σ the electrical conductivity) is the electrical current flux.

(f) Magnetic field induced by current flow, where $\underline{A}$ is the magnetic vector potential satisfying Poisson's vector equation $\nabla^2 \underline{A} = -\mu\underline{i}$, with μ the magnetic permeability. The magnetic induction field vector is $\underline{B} = \nabla x \underline{A}$.

(g) Torsion of shafts, with $\phi = \rho$ is the warping function and $u_1 = -\theta x_1 x_2, \quad u_1 = \theta x_1 x_2, \; u_3 = \theta\phi(x_1, x_2)$ is the displacement field. Furthermore, x_3 is the longitudinal axis of the shaft and θ its angle of twist per unit length.

3.2 Indirect BEM Formulation

For the Cauchy problem involving mixed BC and defined by eqs (3.1)–(3.3), one can express $p(\underline{x})$ as a single–layer potential in the form

$$p(\underline{x}) = \int_S \sigma(\underline{\Xi})\, u(\underline{x},\underline{\Xi}) dS(\underline{\Xi}) + C \qquad (3.16)$$

where σ is the unknown source density and C is a constant to insure uniqueness of solution. If C is omitted, then the supplementary integral condition $\int_S \sigma(\underline{\Xi}) dS(\underline{\Xi}) = 0$ is needed to enforce uniqueness. Through differentiation in the normal direction *n*, we have

$$f(\underline{x}) = \int_S \sigma(\underline{\Xi}) \frac{d}{dn} u(\underline{x},\underline{\Xi}) dS(\underline{\Xi}) \qquad (3.17)$$

The limiting expressions for eqs (3.16) and (3.18) as $\underline{x} \to \underline{X}$ take the form

$$p(\underline{X}) = \int_S \sigma(\underline{\Xi}) u(\underline{X},\underline{\Xi}) dS(\underline{\Xi}) + C \qquad (3.18)$$

$$f(\underline{X}) = -\frac{1}{2}\sigma(\underline{X}) + \int_S \sigma(\underline{\Xi}) \frac{d}{dn} u(\underline{X},\underline{\Xi}) dS(\underline{\Xi}) \qquad (3.19)$$

Note that the fundamental solution has a factor of 4π in the denominator and that the limiting procedure involving S_ε takes place only from inside V to S, thus giving half the total jump factor. Using Eqn. (3.18) for $\underline{X} \in S_1$ and Eqn. (3.19) for $\underline{X} \in S_2$ allows the source density σ to be expressed in terms of the BC. Then, the former equation is solved for $\underline{X} \in S_2$ giving the unknown p, while the latter equation is solved for $\underline{X} \in S_1$ giving the unknown f.

The Dirichlet and Neumann types of problems are special cases of the Cauchy problem. It is also possible to express $p(\underline{x})$ as a double–layer potential of unknown source density, in which case the solution to the Direchlet problem exists and is unique. For the Neumann problem, a sufficient condition for existence of solution of Eqn. (3.19) is that $\int_S \overline{f}(\underline{\Xi}) dS(\underline{\Xi}) = 0$.

3.3 Direct BEM Formulation

The direct BEM employs physical, rather than fictitious, quantities in the formulation. Here we start with Green's second theorem, although other approaches such as the MWR are also possible. We let $u = u(\underline{x}, \underline{\xi})$ be the first fundamental solution, while $\partial u / \partial n = \partial u(\underline{x}, \underline{\xi}) / \partial n(\underline{x}) = u_n(\underline{x}, \underline{\xi})$ is the second fundamental solution. In addition, $v = p(\underline{x})$ and $\partial v / \partial n = \partial p(\underline{x}) / \partial n(\underline{x}) = f(\underline{x})$. Thus,

$$\int_V (u\nabla^2 p - p\nabla^2 u) dV = \int_V p(x)\delta(\underline{x}, \underline{\xi}) dV(\underline{\xi}) + \int_V u\nabla^2 p dV = p(\xi) + \int_V u\nabla^2 p dV =$$

$$= \int_S \{u(\underline{x}, \underline{\xi}) f(\underline{x}) - p(\underline{x})\ u_n(\underline{x}, \underline{\xi})\}\ dS(\underline{x}) \qquad (3.20)$$

which is known as Green's third identity. If $\underline{\xi}$ lies outside V, then $p(\underline{\xi}) = 0$. Taking the limit as $\underline{\xi} \to \Xi$, gives the following BIE:

$$0.5 p(\Xi) + \int_S u_n(X, \Xi) p(\underline{X}) dS(\underline{X}) = \int_S u(\underline{X}, \Xi) f(\underline{X}) dS(\underline{X}) \qquad (3.21)$$

For non-smooth boundaries, the jump term is $c(\Xi)$ and depends on the local curvature.

In solving the Cauchy problem, Eqn. (3.21) is written for $\underline{\Xi} \in S_1$, then for $\underline{\Xi} \in S_2$ and finally the surface integrals are also broken into S_1 and S_2 parts. This allows for determination of p on S_2 and for f on S_1 in terms of the BC. Subsequently, Eqn. (3.21) can be used to determine p in the interior V. If f is also desired in the interior, then differentiation of Eqn. (3.21) in the form $f(\underline{\xi}) = dp(\underline{\xi})/dn(\underline{\xi})$ yields

$$f(\underline{\xi}) = \int_S \left\{ f(\underline{X}) \frac{d}{dn} u(\underline{X}, \underline{\xi}) - p(\underline{X}) \frac{d^2}{dn^2} u(\underline{X}, \underline{\xi}) \right\} dS(\underline{X}) \qquad (3.22)$$

that can be used for that purpose. Finally, if use of the direct BEM gives rise to Fredholm integral equations of the first kind, then problems of non-uniqueness arise and are treated in the same way as in the indirect BEM.

Table 1: Boundary Integral Equation Formulations in Potential Theory

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The Single-Layer Potential:

$$u(\underline{x}) = \iint \rho(\underline{\xi})\, G(\underline{\xi}, \underline{x})\, dS(\underline{\xi})$$

$$q(\underline{x}) = -c(\underline{x})\rho(\underline{x}) + \iint \rho(\underline{\xi}) \frac{\partial G(\underline{\xi}, \underline{x})}{\partial n_x} dS(\underline{\xi})$$

For Dirichlet BC, the first equation is a FIE of the 1st kind in terms of ρ.
For Neumann BC, the second equation is a FIE of the 2nd kind in terms ρ.

The Double-layer Potential:

$$u(\underline{x}) = -c(\underline{x})\mu(\underline{x}) + \iint \mu(\underline{\xi}) \frac{\partial G(\underline{\xi}, \underline{x})}{\partial n_\xi} dS(\underline{\xi})$$

For Dirichlet BC, we have a FIE of the 2nd kind in terms of μ.
For Neumann BC, the equation is hypersingular.

Direct Formulation by MWR:

$$c(\underline{x})u(\underline{x}) + \iint u(\underline{\xi}) \frac{\partial G(\underline{\xi}, \underline{x})}{\partial n_x} dS(\underline{\xi}) = \iint q(\underline{\xi}) G(\underline{\xi}, \underline{x}) dS(\underline{\xi})$$

For Dirichlet BC, we have a FIE of the 1st kind for q.
For Neumann BC, we have a FIE of the 2nd kind for u.

Fundamental Solutions:

For 3D problems, the fundamental solution corresponding to a point source of unit strength is

$$G(\underline{\xi},\underline{x}) = \frac{1}{r(\underline{\xi},\underline{x})} \quad \text{and} \quad c(\underline{x}) = 2\pi$$

while for 2D problems,

$$G(\underline{\xi},\underline{x}) = \ln\frac{1}{r(\underline{\xi},\underline{x})} \quad \text{and} \quad c(\underline{x}) = \pi$$

Note: For Poisson's equation, the volume integral $\iiint g(\underline{z})G(\underline{z},\underline{x})dV(\underline{z})$ must be included in the RHS of all boundary integral equations.

Boundary Integral Equations for Solution in the Interior:

(a) In all cases, the CPV in the integral involving the $\partial G/\partial n$ kernel is irrelevant.
(b) In the indirect formulation, $c(\underline{x}) = 0$.
(c) In the direct formulation, $c(\underline{x}) = 4\pi$ for 3D and $c(\underline{x}) = 2\pi$ for 2D.

3.4 Numerical Implementation

As a vehicle for illustrating the numerical implementation of the boundary integral equations derived in the previous section (which are summarized in Table 1), we will consider the 2D direct formulation given by Eqn. (3.21). At first, the surface of the problem (here the perimeter) is divided into a number of segments (boundary elements), each of which is described by series of nodes. Also, values of the potential $u(\underline{x})$ and of the flux $q(\underline{x})$ are collocated at those same nodes (as in the case of isoparametric finite elements) or at any other set of nodes on the surface (as in the case of sub- or super- parametric finite elements). The simplest possible case is shown in Fig. 4(a) where straight–line elements are defined between two nodes on the surface and $u(\underline{x})$ as well as $q(\underline{x})$ are collocated at the centroid of each element to avoid creation of artificial corners. It is noted that up to this point, the BIE (3.21) is an exact statement for the solution of the corresponding BVP.

By allowing the field point $\underline{x}$ to sequentially coincide with all the collocation nodes, the following system of equations is generated:

$$c(\underline{x}_i)u(\underline{x}_i) + \sum_{j=1}^{N} \int_{S_j} u\left(\underline{\xi}_j\right) F\left(\underline{\xi}_j, \underline{x}_i\right) dS = \sum_{j=1}^{N} \int_{S_j} q\left(\underline{\xi}_j\right) G\left(\underline{\xi}_j, \underline{x}_i\right) dS \qquad (3.23)$$

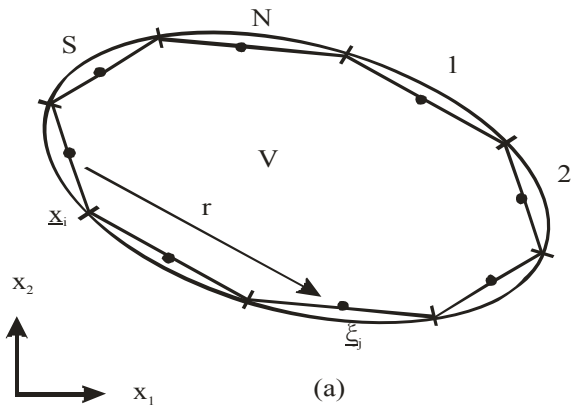
where i, j = 1,2…, N.

We note that integration over the entire surface S has been broken down into integration over sub-surfaces S_j, each corresponding to a boundary element j. It is also reminded that in the above equation, $c(\underline{x}) = \pi$ since the element is a straight line. Also, $G(\underline{\xi}, \underline{x}) = -\ln r(\underline{\xi}, \underline{x})$, while $F(\underline{\xi}, \underline{x}) = -\partial G(\underline{\xi}, \underline{x}) / \partial n_x$. Next, the surface integrals can be evaluated using numerical quadrature, i.e.,

$$\int_{S_j} u(\underline{\xi}_j) F(\underline{\xi}_j, \underline{x}_i) dS = u(\underline{\xi}_j) \int_{S_j} F(\underline{\xi}_j, \underline{x}_i) dS = u(\underline{\xi}_j) \sum_{k=1}^{K} w_k F(\underline{\xi}_j(\eta_k), \underline{x}_i) = u_j F_{ij} \qquad (3.24)$$

and

$$\int_{S_j} q(\underline{\xi}_j) G(\underline{\xi}_j, \underline{x}_i) dS = q(\underline{\xi}_j) \int_{S_j} G(\underline{\xi}_j, \underline{x}_i) dS = q(\underline{\xi}_j) \sum_{k=1}^{K} w_k G(\underline{\xi}_j(\eta_k), \underline{x}_i) = q_j G_{ij} \qquad (3.25)$$



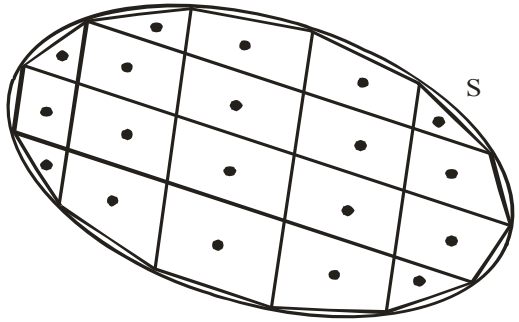


Figure 4: (a) Surface and (b) interior discretization of continuous body *V* with surface *S*

In the above, the fact that both potential and flux remain constant over a boundary element has been taken into account. In terms of notation, η and w are the Gauss points and weights, respectively. The former are normalized, usually as $0 \le \eta \le 1,$ so that functional relationship can be written between $\underline{\xi}_i = (\xi_1, \xi_2)_i$ and η. Details on the derivation of Gaussian-type quadrature formulas can be found elsewhere (Stroud and Secrest, 1966). If suffices to say here that the integrand is expanded in terms of a basis of orthogonal polynomials, whose zeros are the integration points, and the weights are found by integrating the polynomials over the normalized interval of interest.

For nonsingular cases, $(\underline{x} = \underline{\xi})$, Gauss–Legendre quadrature is adequate since the use of K integration points exactly integrates a polynomial of degree *2K-1*. For singular cases $(\underline{x} = \underline{\xi})$, however, the formulas in eqs (3.24) and (3.25) cannot be used because the integrand is discontinuous. For the mild singularity of kernel $G(\underline{\xi}, \underline{x}) = \ln r,$ it is possible to find special integration formulas, such as the log-weighted Gaussian quadature (Stroud and Secrest, 1966). If not, one uses the same approach as for the strongly singular kernel $F(\underline{\xi}, \underline{x})$, whereby the Cauchy principal-value (CPV) definition dictates a subdivision of the boundary element in such a way so as to exclude the singularity. Ordinary Gaussian quadrature can then be used over each sub-element. Also, for simple enough kernels such as those encountered in potential problems and for straight-line boundary elements, the singular integrations can be performed analytically (Jawson and Symm, 1977). In all cases, the final result is the following system of linear algebraic equations:

$$\sum_{i,j=1}^{N} \left(F_{ij} + c_j \delta_{ij}\right) u_j = \sum_{i,j=1}^{N} G_{ij} q_j \qquad (3.26)$$

In the above, δ_{ij} is Kronecker's delta, which is equal to unity if $i = j$ and zero otherwise. In the general case of a non-smooth boundary, the jump terms c_j can be obtained by imposing a uniform potential to the problem in question, in which case the right hand side of Eqn. (3.26) is zero, thus allowing c_j to be obtained as the sum of all the off-diagonal terms of F_{ij}. The above system of equations can now be re-ordered and solved for the N unknown values of the potential and flux in terms of the prescribed boundary conditions.

Once the boundary-value problem has been solved, the interior version of Eqn. (3.23) with $c(\underline{x}_j) = 2\pi$ can be used to find values for the potential in the interior $(\underline{x}_i \in V)$. The numerical process is the same, except that the kernels $G(\underline{\xi}, \underline{x})$ and $F(\underline{\xi}, \underline{x})$ need to be re-evaluated since radial distance $r(\underline{\xi}, \underline{x})$ changes every time a new interior point is specified.

In the case of Poisson's equation, it is necessary to evaluate volume integrals of the form $\iiint g(\underline{z})G(\underline{z}, \underline{x})dV(\underline{z})$, where $g(\underline{z})$ is a prescribed forcing function. To that purpose, the volume area (in a two-dimensional problem) is discretized into cells, as shown in Fig. 4(b). Again, the simplest approach is to use triangles and/or quadrilaterals with collocation taking place at the centroid of each cell. For quadrilaterals, integration over each cell can be accomplished through double application of 1D Gauss-Legendre formulas, i.e,

$$\iint_{V_j} g\left(\underline{z}_j\right)G\left(\underline{z}_j, \underline{x}_i\right)dV = g\left(\underline{z}_j\right)\iint_{V_j} G\left(\underline{z}_j, \underline{x}_i\right)dV = g\left(z_j\right)\sum_{k=1}^{K}\sum_{\ell=1}^{L} w_k w_\ell G\left(z_j\left(\eta_k, \eta_\ell\right), \underline{x}_i\right) \qquad (3.27)$$

In the case of triangles, special quadrature formulas for triangular (or natural) coordinates can be found in the literature (Zienkiewicz, 1977).

Nowadays, curved isoparametric boundary elements with higher order interpolation functions are commonly used in commericial BEM computer programs (Lachat and Watson, 1976). Also, cubic splines have been used to interpolate the field variable (the potential) and its derivatives over any type of element (Prenter, 1989). As an example, consider the quadratic boundary element of Fig. 5. The potential (and similarly the flux) is interpolated as

$$u\left(\underline{\xi}_j\right) = N_1(\eta)u_j^1 + N_2(\eta)u_j^2 + N_3(\eta)u_j^3 = \sum_{m=1}^{3} \mathrm{N}_m(\eta)u_j^m \qquad (3.28)$$

where the shape functions N_m are quadratic functions in η, i.e.,

$$\mathrm{N}_1 = 0.5\eta(\eta-1), \qquad \mathrm{N}_2 = (1-\eta)(1+\eta), \qquad \mathrm{N}_3 = 0.5\eta(\eta+1) \qquad (3.29)$$

The same shape functions are used in mapping the three-noded element from the η-space, where it is a straight line, to the 2D space (x_1, x_2), where the shape varies as a quadratic function of the coordinates. The spatial coordinates are fitted through the element's nodal values as $\underline{x} = (x_1, x_2) = \sum_{m=1}^{3} \mathrm{N}_m(\eta)\underline{\xi}_m$, while the length of the element is given by the determinant of the Jacobian transformation J, i.e.,

$$dS = \left(\left(\frac{dx_1}{d\eta}\right)^2 + \left(\frac{dx_2}{d\eta}\right)^2\right)^{\frac{1}{2}} d\eta = |J(\eta)|d\eta \qquad (3.30)$$

As a result, all integrations in Eqn. (3.23) can now be carried out as follows:

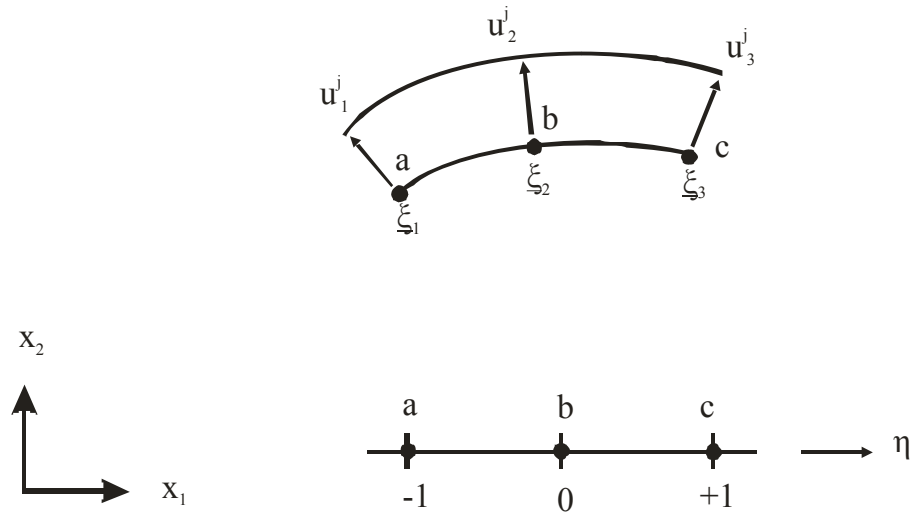


Figure 5: Line element in the physical *(x_1, x_2)* coordinate system and in the intrinsic coordinate system *η*

$$\int_{S_j} u\left(\underline{\xi}_j\right) F\left(\underline{\xi}_j, \underline{x}_i\right) dS = \int_{-1}^{1} \Sigma^3 \mathrm{N}_m(\eta) u_j^m F\left(\underline{\xi}_j(\eta), \underline{x}_i\right) \left|J(\eta)\right| d\eta = \Sigma^3 u_j^m \left(\Sigma^k w_k N_m(\eta_k) F\left(\underline{\xi}_j(\eta_k), \underline{x}_i\right) \left|J(\eta_k)\right| \right) \qquad (3.31)$$

The same principles can be applied for area integrations, where isoparametric quadrilateral boundary elements, which are the 2D extensions of line elements, or triangular boundary elements, which are based on the concept of natural coordinates, can be used. As far as the numerical solution of 3D problems is concerned, the area elements discussed in the context of 2D problems are used for surface integrations, while for volume integrals use is made of brick and tetrahedral elements, which are the 3D extensions of quadrilaterals and triangles (Zienkiewicz, 1977).

3.5 Numerical Example

Consider the 2D steady-state temperature distribution in a circular cylinder of infinite extent (Brebbia et al., 1984). As shown in Fig. 6, the prescribed surface temperature across a cross-section of radius R is $T(r = R, \theta) = T_0$ for $0 \le \theta \le \pi$ and equal to zero for $0 \le \theta \le 2\pi$. The analytical solution involves solving Laplace's equation in polar coordinates *(r, θ)* with *T* identified as *u*, and is given in terms of Fourier series:

$$T(r,\theta)/T_0 = 0.5 + 2.0\,\Sigma_{n=1}^{\infty} \frac{1}{n\pi}\left(\frac{r}{R}\right)^n \sin n\theta, \qquad n = 1,3,5,... \qquad (3.32)$$

The solution obtained by the BEM using two meshes, both composed of straight-line elements, along with the results of the analytical solution, are shown in Table 2 for eight internal nodes. The first and second meshes respectively consist of 24 and 48 equally spaced elements. The sign convention regarding the way the surface is traversed is CCW so as to keep the normal on the RHS of an observer. The first internal point is at the center of the cylinder, while the remaining seven are at *r/R = 0.5* with a spacing of π/6 radians, starting from the x_2 axis. Since there is a discontinuity in the surface temperature distribution at $\theta = 0$ and π, the radial flux $q = \partial T / dn = \partial T / \partial r$ at the surface will also have a discontinuity. This is reproduced in Fig. 7 for the first quadrant of the cylinder.

Table 2: Temperature Distribution at Internal Points

Location	Analytical	BEM-Mesh 1	BEM-Mesh 2
1	0.500	0.500	0.500
2	0.795	0799	0.796
3	0.773	0.776	0.774
4	0.687	0.689	0.688
5	0.500	0.500	0.500
6	0.313	0.311	0.312
7	0.227	0.224	0.226
8	0.205	0.201	0.204

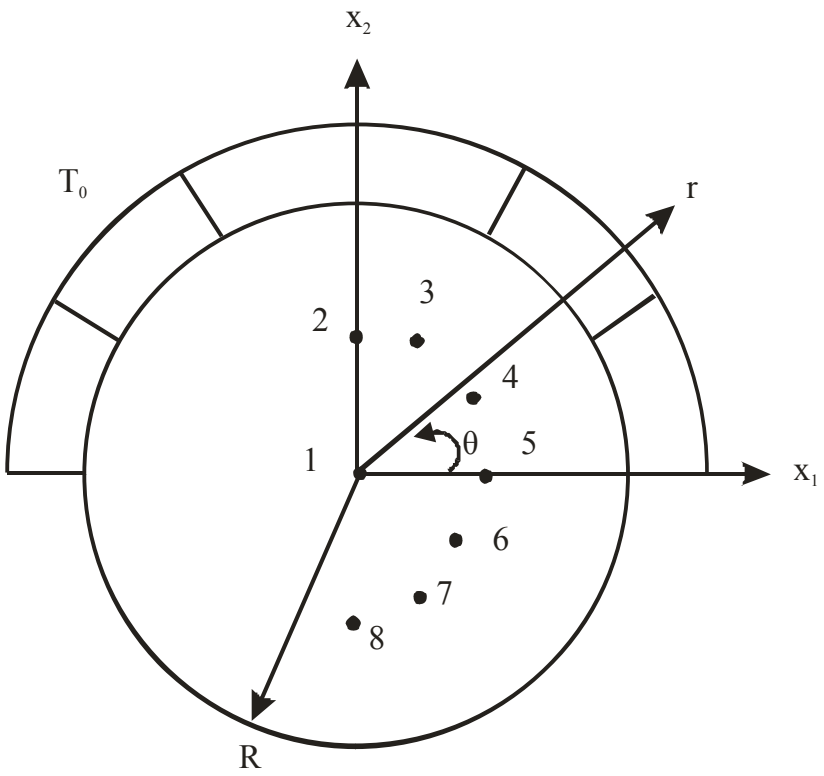


Figure 6: Circular cylindrical disc under constant surface temperature T_0 on the upper boundary

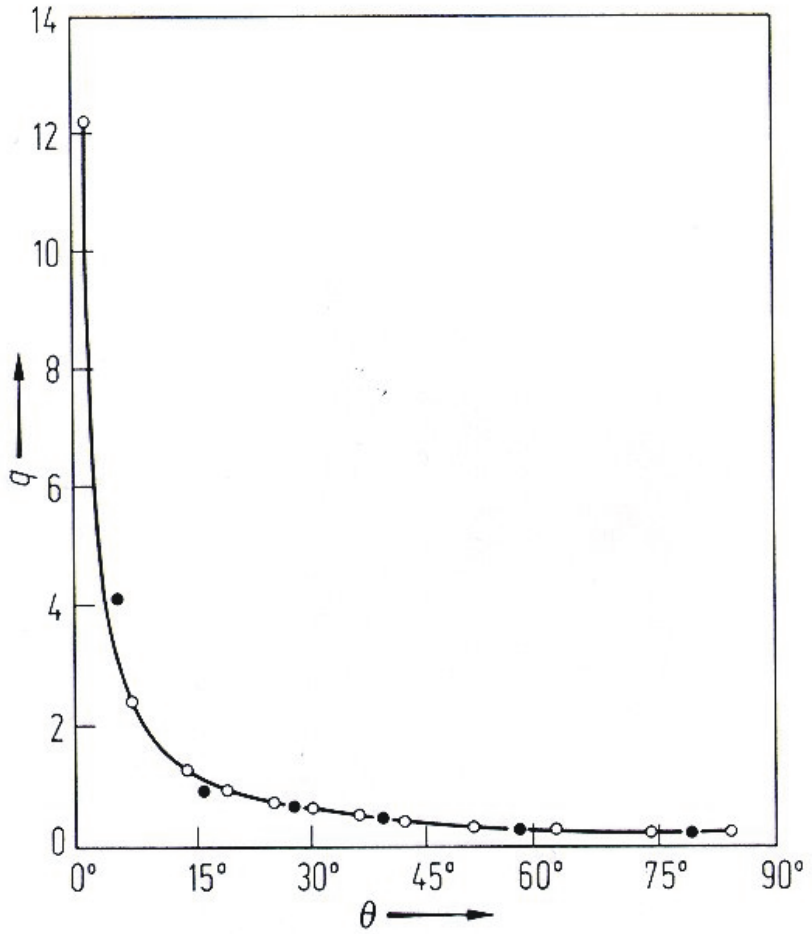


Figure 7: Radial flux q on the surface of the disc versus angle θ (___ = analytical solution, ●●● = mesh no. 1, °°° = mesh no. 2)

4. THE BEM IN ELASTOSTATICS

4.1 Problem Statement

Since many of the concepts regarding boundary integral equations for elastostatics are common with those employed in potential theory, the treatment here will be brief. First, the governing equation for elastostatics is the equilibrium statement involving the stresses σ_{ij} written in terms of the displacement vector $\underline{u}(\underline{x})$, i.e.,

$$\sigma_{ij,j}(\underline{x}) + b_i(\underline{x}) = \mu\, u_{i,kk}(\underline{x}) + \frac{\mu}{1-2\nu} u_{j,ji}(\underline{x}) + b_i(\underline{x}) = 0 \qquad (4.1)$$

The above second order, partial differential equation system is known as Navier's equations. We employ indicial notation with the summation convention implied for repeated indices. Furthermore, commas indicate partial differentiation with respect to the subscripts that follow. Finally, $\underline{b}(\underline{x})$ is the body force, μ is the shear modulus and ν is Poisson's ratio.

The equations of equilibrium are accompanied by kinematic boundary conditions of the form

$$u_i(\underline{x}) = \bar{u}_i(\underline{x}), \qquad \underline{x} \in S_1 \tag{4.2}$$

and by natural boundary conditions of the form

$$t_i(\underline{x}) = \sigma_{ij}(\underline{x}) n_j(\underline{x}) = \bar{t}_i(\underline{x}), \quad \underline{x} \in S_2 \tag{4.3}$$

where t_i are the surface tractions and n_j are components of the outward pointing, unit normal vector on S.

4.2 Boundary Integral Equation Formulation

We will restrict attention to the direct formulation, because it is for more common than indirect formulations. As with potential theory, it is possible to start with a weighted residual statement of the form (Brebbia et al., 1984)

$$R(\tilde{u}_i) = \iiint \left(\tilde{\sigma}_{ij,j} + b_i\right) u_i^* dV - \iint_{S_1} \left(\bar{u}_i - \tilde{u}_i\right) t_i^* dS + \iint_{S_2} \left(\bar{t}_i - \tilde{t}_i\right) u_i^* dS \qquad (4.5)$$

such that $R(\tilde{u}_i) \to 0$ as $\tilde{u}_i \to u_i$, with u_i^* and t_i^* being displacements and tractions corresponding to the weighting field. In elastostatics, however, it is more convenient to use the reciprocity relation

$$\iiint \sigma_{ij} \varepsilon_{ij}^* dV = \iiint \varepsilon_{ij} \sigma_{ij}^* dV \qquad (4.6)$$

which derives from the symmetry of the stress and strain (ε_{ij}) tensors associated with two distinct equilibrium states. The un-starred state is the one for which a solution is sought, while the starred state can then be identified with the generalized Green's function, i.e., with the solution of Eqn. (4.1) for a point body force of unit strength $\underline{b}^*(\underline{x}) = \delta(\underline{\xi} - \underline{x})\, e_i$, where e_i is the unit vector. The boundary conditions here are of the radiation type, so that both $u_i^* = G_{ij}(\underline{\xi}, \underline{x})\, e_j$ and $t_i^* = F_{ij}(\underline{\xi}, \underline{x})\, e_j$ go to zero as $|\underline{\xi} - \underline{x}| \to \infty$.

Integration by parts of both sides of the reciprocity relation and use of the equilibrium condition, along with the definition of small strains as $\varepsilon_{ij} = 0.5\left(u_{i,j} + u_{j,i}\right)$, results in the reciprocal theorem of Betti, given below as

$$\iiint u_i b_i^* \, dV + \iint u_i t_i^* \, dS = \iiint u_i^* b_i \, dV + \iint u_i^* t_i \, dS \qquad (4.7)$$

Further substitution of $b^*(\underline{x})$ and of the fundamental solutions u_i^*, t_i^* gives Somigliana's identity as follows:

$$u_i(\underline{x}) = \iint G_{ij}\left(\underline{\xi}, \underline{x}\right) t_j\left(\underline{\xi}\right) dS\left(\underline{\xi}\right) - \iint F_{ij}\left(\underline{\xi}, \underline{x}\right) u_j\left(\underline{\xi}\right) dS\left(\underline{\xi}\right) +$$
$$+ \iiint G_{ij}\left(\underline{z}, \underline{x}\right) b_j\left(\underline{z}\right) dV\left(\underline{z}\right), \quad \underline{x} \in V \qquad (4.8)$$

The fundamental solution G_{ij} has the form (Rizzo, 1967)

$$G_{ij}\left(\underline{\xi}, \underline{x}\right) = \left\{(3-4\nu)\delta_{ij} + r_{,i}\, r_{,j}\right\} / \left\{16\pi\mu(1-\nu)r\right\} \qquad (4.9)$$

in 3D, where $r = |\, \underline{r}\left(\underline{\xi}, \underline{x}\right)|$ and $r_{,i} = \partial r / \partial x_i = -\left(\xi_i - x_i\right)/r$. The corresponding expression for $F_{ij}\left(\underline{\xi}, \underline{x}\right)$ can be found from processing solution G_{ij}.

This requires processing the displacement–strain relation and the elastic constitutive law $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$, with $C_{ijk\ell}$ the elasticity tensor:

$$C_{ijk\ell} = \frac{2\mu\nu}{1-2\nu}\delta_{ij}\delta_{k\ell} + \mu\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right) \qquad (4.10)$$

Physically speaking, $G_{ij}\left(\underline{\xi},\underline{x}\right)$ and $F_{ij}\left(\underline{\xi},\underline{x}\right)$ respectively represent the displacement and tractions in the *i-th* direction at point $\underline{\xi}$ corresponding to a unit force in the *j-th* direction acting at point $\underline{x}$.

As it stands, the Somigliana identity cannot be used for solving BVP. Instead, a boundary integral equation needs to be produced, by allowing point $\underline{x}$ to ascend to the surface S through a limiting process. As with the potential problem case, kernel $G_{ij}\left(\underline{\xi},\underline{x}\right)$ exhibits a mild (integrable) singularity of the form 1/r, while kernel $F_{ij}\left(\underline{\xi},\underline{x}\right)$ exhibits a strong singularity of the form $\partial(1/r)/\partial r(\partial r/\partial n)$, necessitating a CPV interpretation of the corresponding integral. The relevant manipulations can be found in Rizzo (1967) and thus

$$c_{ij}\left(\underline{x}\right)u_j\left(\underline{x}\right) + \iint F_{ij}\left(\underline{\xi},\underline{x}\right)\,u_j\left(\underline{\xi}\right)dS\left(\underline{\xi}\right) = \iint G_{ij}\left(\underline{\xi},\underline{x}\right)\,t_j\left(\underline{\xi}\right)dS\left(\underline{\xi}\right) +$$

$$+ \iiint G_{ij}\left(\underline{z},\underline{x}\right)\,b_j\left(\underline{z}\right)dV\left(\underline{z}\right), \quad \underline{x} \in S \qquad (4.11)$$

where $c_{ij}\left(\underline{x}\right) = 0.5\delta_{ij}$ in the case of a smooth surface.

4.3 Numerical Implementation

As with the case of potential problems, numerical solution in elastostatics follows four basic steps:

(a) The boundary S is discretized using surface elements, over which the displacements and tractions are interpolated using shape functions. Since we are now dealing with vector quantities, each component of $\underline{u}$ and $\underline{t}$ is interpolated in the way depicted by Eqn. (3.28).

(b) Equation (4.11) is sequentially applied at each node $\underline{x}^i, i = 1,2,...,N1,$ where *N1* is the total number of nodal points. The integrals over $S = \sum_{j-1}^{N2} S_j,$ where *N2* is the total number of surface elements, are computed on an element-by-element basis using numerical quadrature. Again, special techniques must be used in conjunction with singular $\left(\underline{\xi} = \underline{x}\right)$ cases. The final result is an *N1xN1* system of linear algebraic equations like Eqn. (3.26) involving the boundary displacements and tractions.

(c) Next, the appropriate boundary conditions are imposed, thus allowing solution of the unknown boundary values in terms of the prescribed ones. Standard matrix reduction techniques can be used at this stage. Also, if body forces are present, appropriate volume integration will produce a vector that is directly added to the known right-hand-side of the algebraic system of equations.

(d) Once the boundary value problem has been solved, then Eqn. (4.8) can be used to obtain the displacement vector at selected interior points. Furthermore, Eqn. (4.8) can be manipulated to yield expressions for the strains and stresses at interior points.

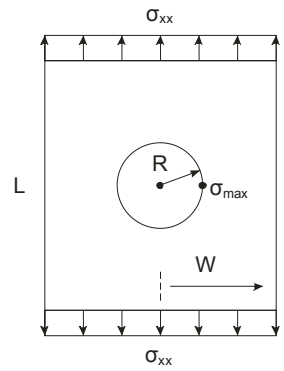
Two additional points that must be kept in mind are the following**:**

(e) In the case of corners in the physical problem, the jump term $c(\underline{x})$ is no longer equal to π in the boundary integral formulation of Eqn. (4.11), but depends on the solid angle (in three dimensions) or the planar angle (in two dimensions) that actually forms at $\underline{x}$. In most numerical implementation schemes, this jump term is measured indirectly through summation of off-diagonal terms in the matrix resulting from the F_{ij} kernel, which is a consequence of the concept of rigid-body motions (Lachat and Watson, 1976).

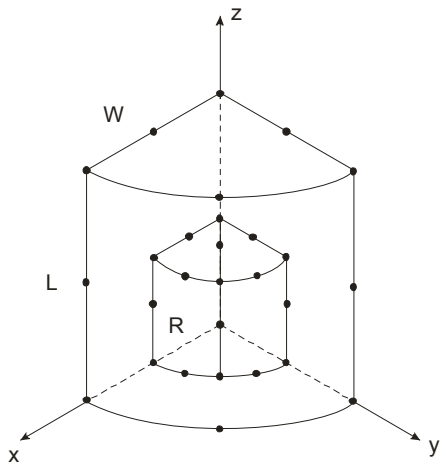
(f) In the case of traction–prescribed only BVP, certain displacement components must be restrained to avoid rigid motions that do not produce any stresses in the body.

4.4 Numerical Example

As an example, consider the problem of stress concentration in a hollow circular cylinder of width $w=5\ in$ and height $L=20\ in$ containing a centered spherical cavity of radius $R=a=3\ in.$, as shown in Fig 8(a). This problem appears in Cruse and Wilson (1978). The cylinder has a Young's modulus $E=18.0\ 10^6\ lb/in^2$ and a Poisson's ratio $v=0.3$ and is subjected to a self-equilibrating uniaxial tension of $\sigma_{xx}=10.0\ 10^3\ lb/in^2$ Due to symmetry, only one-quarter of the cylinder needs to be modeled in the BEM solution, as shown in Fig. 8(b). In particular, two elements (a triangle and a quadrilateral) are used to represent the surface of the cylinder, while four elements (two triangles and two quadrilaterals) are used to represent the surface of the sphere, resulting in a total 28 nodal points. Both these types of elements are of the isoparametric type and employ quadratic shape functions. Figure 9 depicts the numerically obtained stress concentration factor $K = \sigma_{\max}\left(1-(R/w)^2\right)/\sigma_{xx}$ plotted against ratio a/w. Concurrently shown is an experimentally obtained curve, as well as the results from a simpler BEM solution using a total of six linear boundary elements.



(a)



(b)

Figure 8: (a) Circular cylinder of height *L* and width *w* containing a spherical cavity of radius *R=a* under uniform tension and (b) discretization of upper one-quarter of the cavity

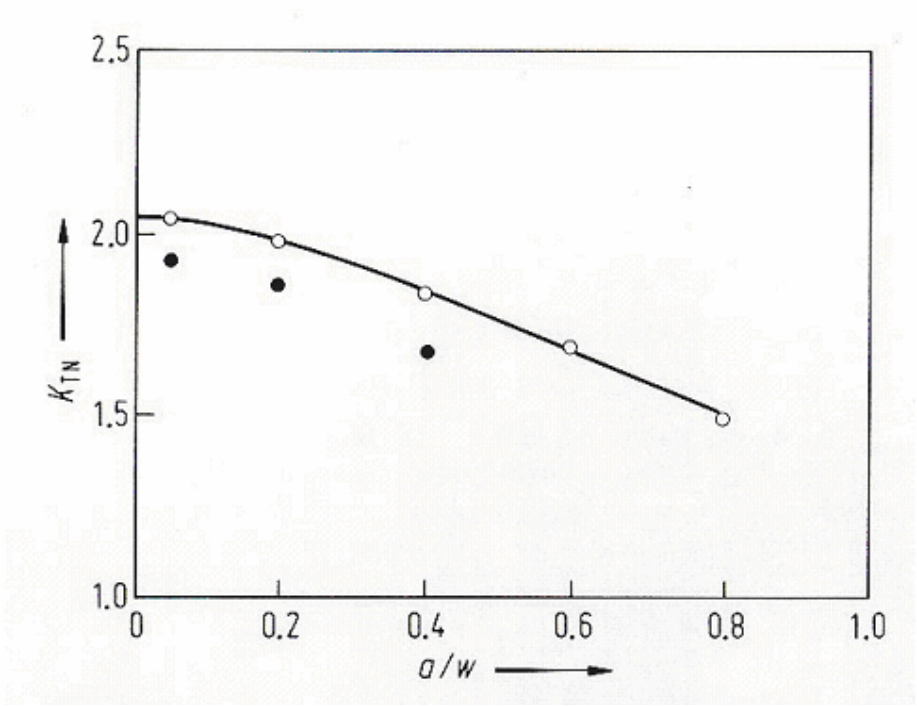


Figure 9: Stress concentration factor *K* versus dimensionless distance *a/w*
(___ = experimental results, ●●● = linear boundary elements, °°° = quadratic boundary elements)

5. THE BEM IN ELASTODYNAMICS

5.1 Introduction

Perhaps surprisingly, the use of integral equation formulations for the analysis of transient phenomena in solids and fluids is over one hundred years old, going back to the Helmholtz-Kirchhoff integral formula, which in turn is the mathematical interpretation of Hyugens' principle (Baker and Copson, 1939). In a large number of cases, part of the boundary is at infinity, which makes the use of integral equation formulations all that more attractive, since they can account for the radiation condition (Morse and Feshbach, 1953; Eringen and Suhubi, 1975; Kupradze 1979).

In here, we will mention some of the earliest developments, whereby integral equation formulations were processed numerically for the solution of BVP. Specifically, we have work by Friedman and Shaw (1962) and Chen and Schweikert (1963) in acoustics and by Banaugh and Goldsmith (1963) in steady-state elastic wave propagation. A milestone paper in computational elastodynamics was that by Cruse and Rizzo (1968) on use of a direct BIEM in conjunction with the Laplace transform for reconstitution of the transient response of the elastic half-plane subject to a surface pulse distribution. In what follows, the basic BIEM formulations will be developed following along the lines given in a comparison study by Manolis (1983).

5.2 Problem Statement

Let V denote the volume and S the bounding surface of an elastic solid. Euclidean *E4* coordinates $(\underline{x},t)$ are employed, and associated with the surface is the outward pointing, unit normal vector $(\underline{n})$. Under the assumption of small displacements and linear elastic, isotropic homogeneous material behavior, the equations of motion written in terms of the displacement vector $u_i(\underline{x},t)$ are as follows:

$$(c_P^2 - c_S^2)u_{i,ij} + c_S^2 u_{j,ii} + f_j = \ddot{u}_j \qquad (5.1)$$

In the above, $f_i(\underline{x},t)$ is the body force per unit mass, while $c_P^2 = (\lambda + 2\mu)/\rho, \quad c_S^2 = \mu/\rho$ are pressure (P) and shear (S) wave speeds in the elastic body. As before, we employ indicial notation with the summation convention implied for repeated indices. Also, commas indicate partial differentiation with respect to the subscripts that follow and dots indicate time derivatives. Finally, λ, μ are the Lame constants, ν is Poisson's ratio and ρ is the mass density.

The displacements and tractions $t_i(\underline{x},t)$ satisfy the following boundary conditions:

$$u_i(\underline{x},t) = \bar{u}_i(\underline{x},t), \quad \underline{x} \in S_1, \qquad t_i(\underline{x},t) = \sigma_{ij}(\underline{x},t) n_j(\underline{x}) = \bar{t}_i(\underline{x},t), \quad \underline{x} \in S_2 \tag{5.2}$$

where σ_{ij} are the stresses and the total surface is subdivided as $S = S_1 \cup S_2$. Furthermore, the displacements and velocities must satisfy initial conditions of the following type:

$$u_i(\underline{x},0) = u_{i0}(\underline{x}), \quad \underline{x} \in V + S, \quad \dot{u}_i(\underline{x},0) = \dot{u}_{i0}(\underline{x}), \quad \underline{x} \in V + S \tag{5.3}$$

Finally, the Sommerfeld radiation condition must hold for outgoing waves.

The Laplace Transform: The direct LT is defined in terms of parameter *s* replacing time *t* reads as follows:

$$\overline{f}(x,s) = L\{f(x,t)\} = \int_0^{+\infty} f(x,t)\exp(-st)dt \qquad (5.4)$$

where the overbar indicates a transformed function. Applied to the hyperbolic PDE (5.1) and taking the initial conditions into account since the transform starts at time *t=0*, the LT yields the following elliptic system of equations:

$$(c_P^2 - c_S^2)\overline{u}_{i,ij} + c_S^2\overline{u}_{j,ii} + \overline{f}_j = s^2\overline{u}_j - su_{jo} - \dot{u}_{jo} \qquad (5.5)$$

The Fourier Transform: The direct FT is defined in terms of frequency ω replacing time *t* reads as follows:

$$\overline{f}(x,\omega) = F\{f(x,t)\} = \int_{-\infty}^{+\infty} f(x,t)\exp(-i\omega t)dt \qquad (5.6)$$

Applied to the hyperbolic PDE (5.1) and ignoring any initial conditions since they are irrelevant to a transform running through the full time axis, the FT yields the following elliptic system of equations:

$$(c_P^2 - c_S^2)\overline{u}_{i,ij} + c_S^2\overline{u}_{j,ii} + \overline{f}_j + \omega^2\overline{u}_j = 0 \qquad (5.7)$$

5.3 Boundary Integral Equation Formulations

It is well known (Wheeler and Sternberg, 1968; Kupradze, 1979) that a system of partial differential equations along with the appropriate boundary and initial conditions may be cast in integral equation form. This holds true for either a system of equations in the time domain or in a transformed (Laplace, Fourier, etc.) domain. Various mathematical questions regarding the validity of such representations in elastodynamics, as well as the existence and uniqueness of solutions that may be obtained, can be found in Sternberg and Eubanks (1955).

The basis for an integral equation formulation in elastodynamics is the dynamic extension of Betti's reciprocal theorem. This theorem is derived from virtual work considerations and essentially relates two different dynamic states for the same elastic body. The obvious choice for one of the elastodynamics states is the unknown state we are seeking to find. A judicious choice for the other state is the appropriate fundamental solution (Green's function) that satisfies the governing equations of motion in region V under radiation conditions only. After some manipulations, the dynamic equivalent to Somigliana's identity can be obtained from Betti's reciprocal theorem, and the limiting form of this identity results in the singular integral equations that form the basis of the BEM.

5.4 Time-domain Integral Equation Formulations

In the time domain, the Green's function is the fundamental point force solution of Stokes, while the singular integral equations are of the form

$$c'_{ij}u'_i(\xi',t) = \int_s \left[G'_{ij}(x',t;\xi') * t'_i(x',t) - F'_{ij}(x',t;\xi') * u'_i(x',t) \right] dS(x') \qquad (5.8)$$

where ξ' and x' are source and receiver points, respectively. In Eqn (5.4), the prime symbol used as a superscript denotes 3D quantities, while superscripts range from 1 to 3. Also, the jump term is

$$c'_{ij} = \delta_{ij} \ for \ \xi' \in V', \quad = 0.5\delta_{ij} \ for \ \xi' \in S', \quad = 0 \ for \ \xi' \notin V' \cup S' \qquad (5.9)$$

If the boundary S' is non-smooth at ξ', c'_{ij} becomes a function of solid angle θ traced at point ξ'. Also, operator * denotes convolution in time:

$$f(x',t) * g(x',t) = \int_0^t f(x',\tau) g(x',t-\tau) d\tau \ for \ t \rangle 0, \quad = 0.0 \ for \ t \le 0 \qquad (5.10)$$

The Green's function G'_{ij} represents the displacement vector at x' and at time t due to a unit impulse in space-time of the form $\delta(t-\tau).(\delta(x'-\xi')$ applied in the three principal directions at ξ' and at time τ, with δ being the Dirac delta function. Usually τ is set equal to zero. Finally, F'_{ij} represents the resulting tractions due to the same unit impulse, and can be obtained from G'_{ij} through the use of the constitutive equations and the definition of tractions. These kernels can be found in Manolis (1983), along with details for the solution procedure.

The approach delineated above is essentially a 3D formulation applied to an infinitely long, cylindrical body whose longitudinal axis coincides with the z-axis of the Cartesian co–ordinate system (see Fig. 10). For a spherical wave assumption emanating from the source point, the formulation corresponds to a plane strain case. The infinitely long cylindrical body may be viewed as a cavity if an interior problem is under consideration.

There is an alternative path for solving 2D problems in the time domain by integral equations. This simply requires the integration of tensors G'_{ij} and F'_{ij} along the longitudinal axis so that their true 2D forms G_{ij} and F_{ij} are obtained. Equation (5.8) then reads as

$$c_{ij}u_i(\xi,t)=\int_0^t \int_S [G_{ij}(x,t;\xi,t_0)t_i(x,t_0)-F_{ij}(x,t;\xi,t_0)u_i(x,t_0)]\,dS(x)dt_0 \qquad (5.11)$$

In the above, S is now the circumference of the 2D body in question. As far as the integration over time is concerned, it appears satisfactory to assume that both u_i and t_i remain constant over time step $[t_0+\Delta t, t_0]$, although linear or quadratic variations have been tried as well.

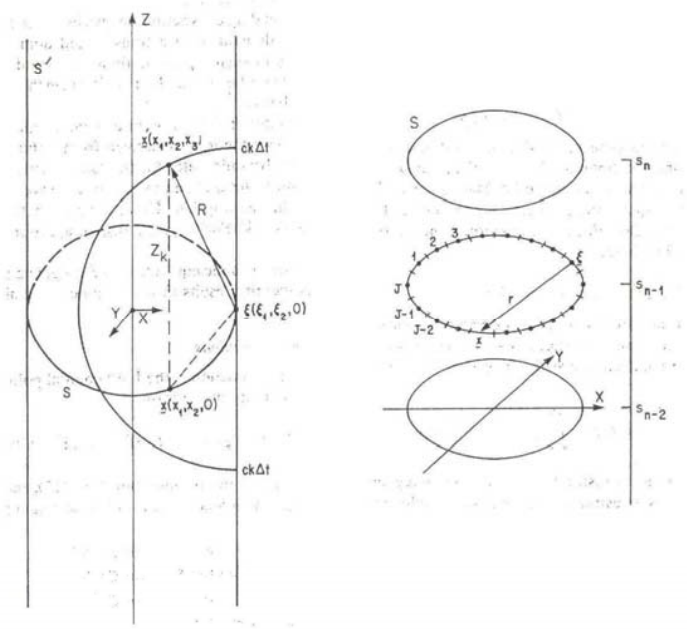


Figure 10: (a) Transient spherical wave from source point ξ; (b) 2D surface *S* at different values of Laplace transform parameter *s*.

5.5 Laplace Transform Domain Integral Equation Formulation

In the Laplace transform domain, the Green's function can be found in Cruse and Rizzo (1968) and the corresponding singular integral equations are of the following quasi-static form:

$$c_{ij}\overline{u}_i(\xi,t) = \int_s \left[\overline{G}_{ji}(x,\xi,s)\bar{t}_i(x,s) - \overline{F}_{ji}(x,\xi,s)\bar{u}_i(x,s) \right] dS(x) \qquad (5.12)$$

The explanatory notes following Eqn (11) apply here as well. It should be reminded at this point bars over a dependent variable denote its Laplace transform. The solution procedure in the Laplace transform domain has been elaborated in Manolis and Beskos (1988) and only a few remarks will be made here.

The problem is essentially static-like for fixed values of the transformation parameter s. The solution procedure consists of solving for the unknown boundary values of $\overline{u}_i$ and $\overline{t}_j$ in terms of the prescribed boundary conditions with the aid of Eqn (5.12), for a spectrum of values of parameter s.

The final step is to numerically invert the transformed solution back to the time domain. The pertinent inversion integral is a contour integral over the complex s-plane in the form

$$f(x,t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \overline{f}(x,s)\exp(st)ds \qquad i=\sqrt{-1} \qquad (5.13)$$

where $\beta\rangle 0$ is arbitrary, but greater than the real part of all singularities of $\overline{f}(x,s)$. Since Eqn (5.13) is an integral over the complex plane, it must be solved for complex values of the parameter s, that is

$$s = \beta + i\left(2\pi/T\right)n \qquad (5.14)$$

In the above, T is the total time interval of interest and $n=1,...N$. Good accuracy of the results cannot be obtained by numerical Laplace inversion algorithms based on real values of s only. The numerical Laplace inversion algorithm that was used in Manolis (1983) was a trapezoidal quadrature scheme for the integral of Eqn (5.9) by Durbin (1974) that employed the fast Fourier transform (FFT) concept of Cooley and Tukey (1965) for expediting the calculations involved.

5.6 Fourier Transform Domain Integral Equation Formulation

As far as the solution in the Fourier transform domain is concerned, Eqn (5.12) can be used if $s = i\omega$, where ω is the frequency of vibration (in *Hz*). The only difference is in the numerical inversion procedure, since the inversion integral is of a different form now:

$$f(x,t) = \int_{-\infty}^{+\infty} \overline{f}(x,\omega)\exp(i2\pi\omega t)d\omega \qquad (5.15)$$

For this case, the FFT algorithm of Cooley and Tukey (1965), which has been extensively used in the past, is employed. Since we are dealing with the one-sided finite transformation, the upper and lower limits in Eqn (5.15) are replaced by 0 and Ω (the maximum frequency of interest). Then, the well-know problem associated with finite Fourier transform, namely the approximation a non-periodic motion by a periodic one, arises. This problem is circumvented by adding a sufficient number of trailing zeros to the transformed solution.

Finally, one should be aware of a difficulty associated with the Fourier transform solution. Specifically, Eqn (5.15) fails to produce a solution for an exterior (or interior) problem for a particular set of frequencies ω corresponding to the eigenvalues of the associated interior (or exterior) problem. The original BVP possesses, of course, a unique solution at those frequencies. This problem was noticed by workers in the field of acoustics and a number of methods were introduced to combat it, such as the technique of Schenk (1968). It also possible, however, for problems in elastodynamics to isolate the eigenvalues of the associated problem by methods such as the one described in Tai and Shaw (1974) and then to modify the solution procedure of the original problem as in acoustics so as to avoid solution divergence.

5.7 Numerical Implementation Aspects

It is obvious that BIE such as Eqn (5.8) and (5.12) cannot, in general, be solved analytically and therefore resort must be made to numerical methods of solution. The bounding surface S (circumference) of the 2D body is therefore discretized into J segments with a node defined at the midpoint of a given segments, as shown in Fig. 10. The spatial variation of the displacements and traction over a given segment is assumed to be constant, although it is possible to define or higher order variations using linear or quadratic isoparametric elements.

In the 3D approach of the time domain formulation, the convolution integral is discretized using discrete time steps so Eqn (5.8) is rewritten as

$$\begin{aligned} 0.5\delta_{ij}u_i(\xi,t) = \sum_{K=1}^{N}\int_S \Big[U_{ij}(x,K\Delta t;\xi)t_i(x,(N-K+1)\Delta t\Big) \\ - n_m\Big[T_{ijm}(x,K\Delta t;\xi) + Q_{ijm}(x,K\Delta t;\xi))u_i(x,(N-K+1)\Delta t\Big) \\ - Q_{ijm}\big(x,K\Delta t;\xi)u_i(x,(N-K)\Delta t))\Big]dS(x) \end{aligned} \qquad (5.16)$$

The time axis is now formally discretized into L segments so that total time of interest is $T = L\Delta t$ and a constant temporal value of the displacements and tractions over an interval is adopted. Eqn (5.12) can be recast in the following convenient for numerical computations form:

$$0.5u_j^n = \sum_m\sum_i DG_{ij}^{nm}t_i^m - \sum_m\sum_i DF_{ij}^{nm}u_i^m \qquad (5.17)$$

In the above, the spatial indices range as i, j=1,2,…,J while the temporal indices first sweep as n=1,2,…,L and then as m = 1, 2,…, n. The discrete system matrices DG_{ij}^{nm} and DF_{ij}^{nm} are the spatial integrals of the kernels U_{ij}, T_{ijm} and Q_{ijm}, respectively. In the latter case, use was made of the finite difference scheme that allowed the velocity vector to be expressed terms of the displacement vector.

In the 2D approach of the time domain formulation, Eqn (5.11) can also be recast in the form of Eqn (5.17), where DG_{ij}^{nm} and DF_{ij}^{nm} are now given by the following space-time integrals:

$$DG_{ij}^{nm} = \int_{(m-1)\Delta t}^{m\Delta t} \int_{\Delta s} G_{ij}\, dS dt_0 \qquad (5.18)$$

$$DF_{ij}^{nm} = \int_{(m-1)\Delta t}^{m\Delta t} \int_{\Delta s} F_{ij}\, dS dt_0 \qquad (5.19)$$

In the above formulas ΔS is the length of segment i. Essentially, the difference between the two aforementioned time approaches is that in the former (3D) the integration is done over area segments that move with time, while in the latter (2D) the integration is done over a line segment and then over a time step.

The advantage of using a 3D time domain formulation is that singularities in the kernels are encountered only once in the time marching scheme, when $n = m = 1$ and for $i = j$. These singularities are discussed in Cole et al. (1978) for the anti-plane strain case. If a 2D time formulation were to be used, then a singularity would be encountered every time step m, when i= j. This comment becomes more obvious in reference to Fig. 10.

In the integral transform formulations (Laplace and Fourier), as described in Manolis and Beskos (1981), a static-like system of equations results in marked contrast to the time-domain formulations. Specifically,

$$0.5\overline{u_j} = \sum_i D\overline{G_{ji}}\,\overline{t_i} - \sum_i D\overline{F_{ji}}\,\overline{u_i} \qquad (5.20)$$

where the spatial integration schemes are similar to those used in elastostatics. Eqn (5.20) is therefore solved for a number of values of the transformed parameter s or the frequency ω. The additional price that is paid is that a numerical inverse transformation is required to bring the tractions and the displacements back to the time domain from the transformed spectrum where they are initially produced. The singularities encountered in these formulations, when $i = j$ are of the form $\ln(r)$ for the $\overline{G_{ij}}$ tensor and of the form $r_{,i}n_i/r$ for the $\overline{F_{ij}}$ tensor. The behavior of these singularities as $r \to 0$ is well documented from elastostatics (Brebbia et al., 1984).

5.8 Numerical Example

As an example, consider a P-wave sweeping a circular cylindrical cavity in an otherwise infinitely extending elastic medium (see Fig. 11(a)). This is a plane strain problem and the P-wave has a sharp normal front (since H is the Heaviside function), past which a load tensor with components $\sigma_{xx} = \sigma_0 H(t - t_n)$, $\sigma_{yy} = (\nu/(1-\nu))\sigma_0 H(t - t_n)$, $\tau_{xy} = 0$ envelopes the cavity wall. Time $t = 0$ is the instant the wave first impinges on the cavity, and t_n is the time required for the wave traveling with velocity c_P to reach a given station *n* on the circular boundary.

Several semi-analytical solutions exist (Baron and Matthews, 1961; Pao and Mow, 1972) and the resulting stress distribution is obtained by superimposing the incident wave field described above, to the one scattered by the cavity, and imposing a traction-free condition along the cavity boundary. This latter problem is solved by the BEM. Consider granite with wave velocities $c_P = 208 \times 10^3\ in/\sec$, $c_S = 120 \times 10^3\ in/\sec$ containing a cavity of radius $R = 212\ in$ (*1in=2.54cm*). The time required for the P-wave to envelop the cavity is $2R/c_P = 0.002\ \sec$, while the stress intensity is normalized to $\sigma_0 = 1.0\ lb/in^2$.

The results in Fig. 11(b) are taken from Manolis (1983) and depict the stress concentration factor (SCF) time history at the top (crown) of the cavity, at an angle of 90^0 from the line perpendicular to the wave front.

Three BEM formulations were employed, namely the time-stepping BEM, the Laplace transformed BEM and the time-harmonic BEM with Fourier synthesis. In all cases, the spatial discretization was relatively simple: only 20 isoparametric (i.e., three-noded) surface elements were required for the cavity surface. The total time interval of interest was ten half-transit times ($0.010\,\mathrm{sec}$), so as to match the semi-analytical solutions. Regarding the three BEM formulations, the time-stepping one used 25 time steps, the Laplace transformed one used only 10 values for the Laplace transform parameter and the Fourier transformed one used 32 values for the frequency parameter. Accuracy was comparable, but regarding efficiency, the Laplace transformed BEM performed best.

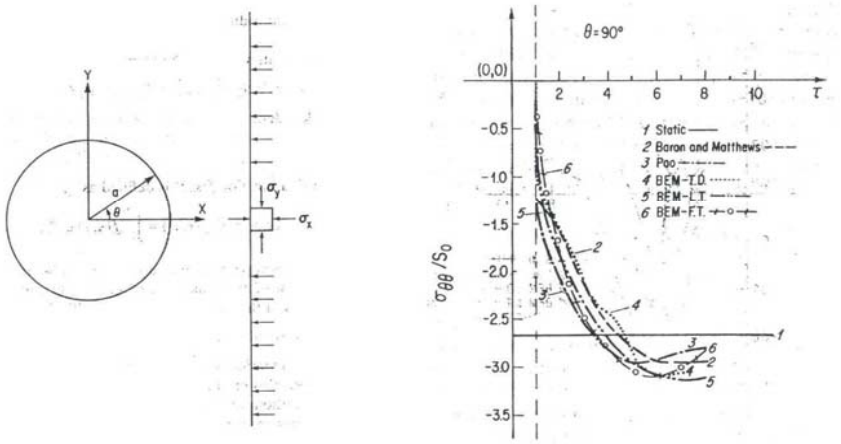


Figure 11: (a) Circular cavity in infinite medium swept by P-wave; (b) stress concentration time history at the crown of the cavity.

6. FUTURE DEVELOPMENTS

Most new developments regarding the BEM will come from the field of software development. Despite the proven effectiveness of the BEM in dealing with continuous systems, and the existence of much in-house computer programs, not much commercial software is available in general. The steps that have to be taken in this respect will follow those already taken in conjunction with the FEM, namely:

(a) Automatic mesh generation schemes, (b) adaptive mesh refinement based on h-type convergence, p-type convergence and their combination, (c) development of both general-purpose as well as special-purpose programs, (d) software development in new languages (e.g., Fortran 90, C^{++}) so as to take advantage of vectorization and parallel processing capabilities of modern computers, and (e) modular-type programming and use of symbolic-type mathematics for solution of relative simple problems using integral equations.

As far as developments on the methodology itself are concerned, the following areas can be identified:

(a) Construction of exact or even approximate fundamental solutions for certain classes of problems (e.g., non-homogenous media, gradient-type constitutive laws) which have been proven difficult for the BEM to track, (b) hybridization of the FEM and BEM as well as the construction of new types of elements (e.g., finite super-elements), (c) development of domain-decomposition techniques for the treatment of large-scale problems, (d) development of mesh-less types of methods whereby integrations are carried out around nodal points dispensing with the need to use elements, and (e) new developments in the conversion of volume integral into surface ones for the efficient analysis of nonlinear problems and (e) refinements in the solution algorithms used currently, especially for transient (time-stepping) problems.

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