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# IMPLEMENTATION OF A NON-SPLITTING FORMULATION OF PERFECT MATCHING LAYER IN A 3D – 4<sup>th</sup> ORDER STAGGERED-GRID VELOCITY-STRESS FINITE-DIFFERENCE SCHEME

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## SUMMARY

One of the most usual and important problems in numerical methods is the truncation of the computational space by artificial boundaries, since such boundaries often produce spurious reflections, polluting the results with artificial noise. Among the various techniques, the Perfect Matching Layers (PML) approach seems to be the most efficient method for the implementation of artificial boundaries at the edges of the computational models. [Wang & Tang 2003] presented a Non-Splitting formulation of PML (NPML) based on the introduction of small perturbations in the wavefield and applied their approach for the cylindrical coordinate system. In this work the NPML technique in Cartesian coordinates and its implementation in a 3D 4<sup>th</sup>-order staggered-grid finitedifference scheme is elaborated. The accuracy and the efficiency of NPML in simple models of media are numerically examined. The results of this PML implementation are compared with the corresponding results calculated with the same finite-difference scheme but with the use of alternative non-reflecting boundaries. Moreover, different tests are performed for examining the implication of the attenuating function and PML thickness relation in the overall PML performance. In general, PML performance is poorer, as expected, at grazing incidence angles but the power of spurious reflections is significantly lower in comparison with the alternative nonreflecting boundaries examined in this work. Large attenuating function coefficient values can produce strong reflections, whereas small values may lead to insufficient attenuation of the impinging waves. An alternative is to use a thicker PML zone, which significantly improves their performance at the cost of increased memory and processing demand. The results show that in order to achieve an optimum memory usage while maintaining PML efficiency, the PML thickness and attenuating function coefficient values need to be optimized for each examined simulation and model.

### INTRODUCTION

The truncation of the computational space by artificial boundaries is one of most important problems in numerical methods. The techniques proposed so far for artificially truncate the computational space are the absorbing boundary conditions (ABCs) and absorbing layers. Among the variations of absorbing layers is the Perfect Matching Layer (PML) technique, which was originally introduced by [Berenger, 1994] in 2-D, and later in 3-D [Berenger, 1996] time domain electromagnetic (EM) numerical simulations and has the property of being perfectly matched with the bulk medium of the computation area.

In this paper the Non-Splitting formulation of PML (NPML) technique introduced by [Wang & Tang, 2003] is followed. The implementation in Cartesian coordinates is elaborated and the accuracy and efficiency of NPML in simple models of media are numerically examined. The results of the NPML implementation are compared with the corresponding results calculated with the same finite-difference scheme but with the use of alternative non-reflecting boundaries. Moreover, different tests are performed for examining the implication of the attenuating function and PML thickness relation in the overall PML performance.

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#### NON - SPLITTING FORMULATION (NPML)

The Non-Splitting formulation is essentially based on a coordinate stretching technique. Assume  $p \in \{x, y, z\}$ ;  $\tilde{p}$  and  $s_p$  are defined in equation (1).

$$\frac{\partial}{\partial \tilde{p}} = \frac{1}{s_p} \frac{\partial}{\partial p}, \quad s_p = 1 + \frac{\Omega_p}{i\omega} \tag{1}$$

Functions  $s_p$  are inversely proportional to frequency, functions  $\Omega_p$  are the attenuating functions in the PML zone. After coordinate stretching, the modified nabla operator is given by equation (2):

$$\tilde{\nabla} = \hat{x} \frac{1}{s_x} \frac{\partial}{\partial x} + \hat{y} \frac{1}{s_y} \frac{\partial}{\partial y} + \hat{z} \frac{1}{s_y} \frac{\partial}{\partial z}$$
(2)

and correspondingly the equation of motion becomes :

$$\rho \frac{\partial \vec{V}}{\partial t} = \tilde{\nabla} \cdot \vec{\tau} \tag{3}$$

Expanding the previous equation and applying the Fourier Transformation results in equation in the frequency domain:

$$-i\omega\rho\vec{V}(\omega) = \begin{vmatrix} \frac{1}{s_x}\tau_{xx,x}(\omega) + \frac{1}{s_y}\tau_{yx,y}(\omega) + \frac{1}{s_z}\tau_{zx,z}(\omega) \\ \frac{1}{s_x}\tau_{xy,x}(\omega) + \frac{1}{s_y}\tau_{yy,y}(\omega) + \frac{1}{s_z}\tau_{zy,z}(\omega) \\ \frac{1}{s_x}\tau_{xz,x}(\omega) + \frac{1}{s_y}\tau_{yz,y}(\omega) + \frac{1}{s_z}\tau_{zz,z}(\omega) \end{vmatrix}$$
(4)

Applying the inverse Fourier Transformation on equation (1) and taking under consideration equation (1)

$$\mathcal{F}^{-1}\left(\frac{1}{s_p}\right) = \mathcal{F}^{-1}\left(1 - \frac{\Omega_p}{\Omega_p + i\omega}\right) = \delta(t) - \Omega_p e^{-\Omega_p t}$$
<sup>(5)</sup>

we obtain equation

$$\mathcal{F}^{-1}\left(\frac{\partial}{\partial \tilde{p}}\right) = (1+\varphi_p)\frac{\partial}{\partial p}, \quad \varphi_p = -\Omega_p e^{-\Omega_p t} *$$
(6)

In conclusion the equation of motion in time domain for NPML is given by :

$$\rho \frac{\partial \vec{V}}{\partial t} = \begin{pmatrix} (1+\varphi_x)\tau_{xx,x} + (1+\varphi_y)\tau_{yx,y} + (1+\varphi_z)\tau_{zx,z} \\ (1+\varphi_x)\tau_{xy,x} + (1+\varphi_y)\tau_{yy,y} + (1+\varphi_z)\tau_{zy,z} \\ (1+\varphi_x)\tau_{xz,x} + (1+\varphi_y)\tau_{yz,y} + (1+\varphi_z)\tau_{zz,z} \end{pmatrix}$$
(7)

The time derivative of Hook's law for elastic media is given by equation (8):

$$\frac{\partial \vec{\tau}_{ij}}{\partial t} = \mathbf{c}_{ijkl} \frac{\partial \vec{\varepsilon}_{kl}}{\partial t}, \quad i, j, k, l \in \{x, y, z\}$$
(8)

 $\vec{\tau}(x, y, z, t)$  is the stress tensor,  $\vec{\varepsilon}(x, y, z, t)$  is the strain tensor and **c** is the stiffness matrix. In frequency domain equation (8) becomes:

$$-i\omega\vec{\tau}_{ij}(\omega) = -i\omega\mathbf{c}_{ijkl}\vec{\varepsilon}_{kl}(\omega), \quad i, j, k, l \in \{x, y, z\}$$
(9)

where  $i\omega$  comes from the time derivative in frequency domain and  $\vec{\tau}_{ij}(\omega)$ ,  $\vec{\varepsilon}_{kl}(\omega)$  are stress and strain tensors in frequency domain. The relation between strain and velocity field is given by equation:

$$-i\omega\vec{\varepsilon}_{kl}(\omega) = \frac{1}{2}(\vec{v}_{k,l}(\omega) + \vec{v}_{l,k}(\omega)), \quad k,l \in \{x, y, z\}$$
(10)

where  $\vec{v}_{k,l}(\omega)$  is the spatial derivative of the velocity vector in the frequency domain.

By applying the coordinate stretching technique according to equation (1), equation (9) becomes:

$$-i\omega\vec{\tau}_{ij}(\omega) = -i\omega\mathbf{c}_{ijkl} \begin{pmatrix} \frac{1}{s_x}\frac{\partial u(\omega)}{\partial x} & \frac{1}{2}\left(\frac{1}{s_x}\frac{\partial v(\omega)}{\partial x} + \frac{1}{s_y}\frac{\partial u(\omega)}{\partial y}\right) & \frac{1}{2}\left(\frac{1}{s_x}\frac{\partial w(\omega)}{\partial x} + \frac{1}{s_z}\frac{\partial u(\omega)}{\partial z}\right) \\ \frac{1}{2}\left(\frac{1}{s_y}\frac{\partial u(\omega)}{\partial y} + \frac{1}{s_x}\frac{\partial v(\omega)}{\partial x}\right) & \frac{1}{s_y}\frac{\partial v(\omega)}{\partial y} & \frac{1}{2}\left(\frac{1}{s_y}\frac{\partial w(\omega)}{\partial y} + \frac{1}{s_z}\frac{\partial v(\omega)}{\partial z}\right) \\ \frac{1}{2}\left(\frac{1}{s_z}\frac{\partial u(\omega)}{\partial z} + \frac{1}{s_x}\frac{\partial w(\omega)}{\partial x}\right) & \frac{1}{2}\left(\frac{1}{s_z}\frac{\partial v(\omega)}{\partial z} + \frac{1}{s_y}\frac{\partial w(\omega)}{\partial y}\right) & \frac{1}{s_z}\frac{\partial w(\omega)}{\partial z} \end{pmatrix}$$
(11)

After applying the inverse Fourier Transformation on the previous equation and rearranging we get:  $\begin{bmatrix}
 ( a_{11}, a_{22}, a_{23}) \\
 ( a_{23}, a_{23}) \\
 ($ 

$$\frac{\partial \vec{\tau}_{ij}}{\partial t} = \frac{\partial}{\partial t} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & \frac{\partial w}{\partial z} \end{pmatrix} \\ + \mathbf{c}_{ijkl} \begin{pmatrix} \phi_x \frac{\partial u}{\partial x} & \frac{1}{2} \left( \phi_x \frac{\partial v}{\partial x} + \phi_y \frac{\partial u}{\partial y} \right) & \frac{1}{2} \left( \phi_x \frac{\partial w}{\partial x} + \phi_z \frac{\partial u}{\partial z} \right) \\ \frac{1}{2} \left( \phi_y \frac{\partial u}{\partial y} + \phi_x \frac{\partial v}{\partial x} \right) & \phi_y \frac{\partial v}{\partial y} & \frac{1}{2} \left( \phi_y \frac{\partial w}{\partial y} + \phi_z \frac{\partial v}{\partial z} \right) \\ \frac{1}{2} \left( \phi_z \frac{\partial u}{\partial z} + \phi_x \frac{\partial w}{\partial x} \right) & \frac{1}{2} \left( \phi_z \frac{\partial v}{\partial z} + \phi_y \frac{\partial w}{\partial y} \right) & \phi_z \frac{\partial w}{\partial z} \end{pmatrix} \end{pmatrix}$$
(12)

Finally the time derivative of the stress-strain relation for the NPML formulation is given by equation:

$$\frac{\partial \vec{\tau}_{ij}}{\partial t} = \mathbf{c}_{ijkl} \frac{\partial (\vec{\varepsilon}_{kl} + \delta \vec{\varepsilon}_{kl})}{\partial t}, \ i, j, k, l \in \{x, y, z\}$$
(13)

Functions  $\phi_p$  (p=x,y,z) are convolutional operators. Convolution of these functions with the partial derivatives of stress or strain e.g. equation (14) for stress, leads in the calculation of the integral in equation or in equation (15) for discrete time levels, which is the scalar notation of convolution between two functions.

$$\begin{split} \varphi_{p} \frac{\partial \tau_{pp}}{\partial p} &= -\Omega_{p} e^{-\Omega_{p}t} * \frac{\partial \tau_{pp}}{\partial p} \Rightarrow P_{pp} = -\Omega_{p} \int_{0}^{\infty} e^{-\Omega_{p}(t-\tau)} \frac{\partial \tau_{pp}(p,\tau)}{\partial p} d\tau \\ (14) \\ P_{pp}^{m} &= -\Omega_{p} \int_{0}^{m\Delta t} e^{-\Omega_{p}(m\Delta t-\tau)} \frac{\partial \tau_{pp}(p,\tau)}{\partial p} d\tau \end{split}$$
(15)

The NPML concept is described in equations (7) and (13). [Wang & Tang, 2003] suggested that it can be expressed as small perturbations in the equation of motion and Hook's law. This is true for equation (12) which can be written in a more compact formulation (equation (13)) but not for equation (7) which cannot be written as the suggested equation (A-12) in [Wang & Tang, 2003]. Nonetheless the system of equations (7) and (13) can be used to describe a homogeneous set of equations for the bulk medium and the PML layer, with the same properties and accuracy described in [Wang & Tang, 2003].

## IMPLEMENTATION OF NPML IN A 3D–4<sup>th</sup> ORDER STAGGERED-GRID VELOCITY-STRESS FINITE-DIFFERENCE SCHEME

The Non Splitting formulation of PML was incorporated in the 3D–4<sup>th</sup> order staggered-grid velocity-stress finitedifference scheme proposed by [Moczo et al., 2002]. The scheme takes into account heterogeneity of the medium by applying volume harmonic and arithmetic averaging of elastic moduli and densities and treats efficiently free surface by using AFDA technique proposed by [Kristek et al., 2002]. Also the anelastic behavior of the medium is taken into account with the use of anelastic functions and anelastic coefficients as introduced in [Kristek and Moczo, 2003].

Merging a 4<sup>th</sup> order finite difference scheme with a 2<sup>nd</sup> order NPML formulation may affect the accuracy and the stability of the resulting scheme. Nevertheless all tests conducted did not show any instability in the results and the accuracy of the resulting scheme was not downsized. The implementation of NPML formulation also introduces the convolution integral shown in equation (15). In the present work this integral was calculated numerically, following the same approximation in Wang and Tang (2003) (trapezoidal rule, e.g. [Davis and

Rabinowitz, 1975]). Trapezoidal rule is of  $2^{nd}$  order accuracy and its application on equation (15) leads to equation (16) which is transformed in a suitable form for incorporating it in a finite difference scheme.

$$P_{pp}^{m} = e^{-\Omega_{p}\Delta t} P_{pp}^{m-1} - \frac{1}{2} \Omega_{p}\Delta t \left| e^{-\Omega_{p}\Delta t} \frac{\partial \tau_{pp}^{m-1}}{\partial p} + \frac{\partial \tau_{pp}^{m}}{\partial p} \right|$$
(16)

In equation (16)  $\Omega_p$  are the attenuating functions inside the PML layer. In the medium they are equal to zero so the system of equations for velocity stress NPML scheme is reduced to the original system of FD equations. Different types of functions have been proposed by different authors for describing  $\Omega_p$  functions. In this paper a modified version of the function proposed by [Collino and Tsogka, 2001] with some slight modifications was used (equation (17)).

$$\Omega_p = \Omega_0 \left(\frac{p}{\delta}\right)^2, \quad \Omega_0 = \log \frac{1}{R} \frac{\tau V}{\delta} \tag{17}$$

where  $\delta$  is the depth of the layer, V is the medium velocity,  $\tau$  is a tuning variable and R the theoretical reflection coefficient. The theoretical reflection coefficient describes the desirable minimum reflection at the PML interface and is a function of the depth of the layer ([Collino and Monk, 1998a]). In this work P wave velocity,  $V_p$ , was used and  $\tau/R$  was set equal to 1.5.

The complete 3D–4<sup>th</sup> order staggered-grid velocity-stress finite-difference scheme for the Non Splitting formulation of PML is given in Appendix. This formulation shows the same accuracy with the originally introduced Splitting formulation of PML (SPML) (e.g. [Festa and Nielsen, 2003], [Marcinkovich and Olsen, 2003])but is much easier to implement because it does not require the field splitting. In terms of computational efficiency NPLM is similar to SPML, but in memory storage is slightly less efficient than SPML since more variables have to be kept in core memory.

#### NUMERICAL TESTING OF NON SPLITTING PML FORMULATION

For testing the accuracy of NPML formulation, the FD synthetics were compared with synthetics calculated using the Discrete Wavenumber method (DWN) ([Bouchon, 1981]; computer code Axitra [Coutant, 1989]). The comparisons were made for a homogeneous space and calculations of synthetics were performed for two different values of Poisson ratio,  $V_p/V_s$ =4 and  $V_p/V_s$ =1.78 (model A and B respectively) in order to examine the stability of the scheme in models with high velocity contrasts. For both types of models, an FD calculation was performed for an enlarged version of each model, with the same source - receiver geometry but located much further from the borders in order to have a reference solution.



Figure 1.a) Configuration of the source-receivers' geometry for the incidence angle test. With the black circles the position of receivers and with the black star the position of the source are denoted respectively. Receivers which all recordings are used in figures are denoted with light gray squares . b) Configuration of the source-receivers' geometry for the incidence angle test in the enlarged model. The colors and symbols used are the same as in figure 1a.

The efficiency of PML for different incidence angles of the wavefield in these models was tested. In Table 1 the parameters of the models used for the two different values of Poisson ratio are shown. The number of grid cells in the x, y, and z directions were MX=300, MY=300, and MZ=500, the spatial grid spacing was h=30 m and the time step for model A dt=0.006 s and for model B dt=0.013 s, respectively. For the enlarged models the number

of grid cells in the x, y, and z directions were MX=350, MY=300, MZ=500. The spatial grid spacing and the time steps for enlarged models (A and B) were the same.

Table 1. Model parameters of the homogeneous model for two different values of Poisson ratio (Model A, 4; Model B, 1.78).  $\alpha$  is the P wave velocity,  $\beta$  the S wave velocity,  $\rho$  is the density and  $Q_p$  and  $Q_s$  the quality factors for P and S waves respectively.

	α (m/s)	β (m/s)	$\rho$ (Kg/m <sup>3</sup> )	$Q_{\rm p}$	$Q_{\rm s}$
Model A	2248	562	2000	1000	1000
Model B	1000	562	2000	1000	1000

A double-couple point source was used for the FD computations. The source was simulated using a body-force term by a method suggested by [Frankel, 1993] and adapted for a staggered-grid by [Graves, 1996]. A Gabor signal,  $s(t) = \exp\left\{-\left[\omega(t-t_s)/\gamma\right]^2\right\} \cos\left[\omega(t-t_s)+\theta\right]$ , was used as a source time function. Here,  $\omega = 2\pi f_p$ ,  $t \in [0,2t_s]$ ,  $f_p$  is the predominant frequency,  $\gamma$  controls the width of the signal,  $\theta$  is the phase shift, and  $t_s$  defines the signal peak position. The parameters of the source for the incidence angle test are given in Table 2.

Table 2. Source parameters for the two configurations correspond to Figures 1a and 1b.  $M_{\theta}$  is the scalar seismic moment,  $\varphi_s$  is the strike,  $\delta$  is the dip,  $\lambda$  is the rake,  $\gamma$ ,  $f_p$ ,  $\theta$  and  $t_s$  are parameters of the Gabor signal.

	$M_{\theta}$ (Nm)	$\varphi_{\rm s}$ (deg)	$\delta$ (deg)	$\lambda$ (deg)	γ	$f_{\rm p}$	θ	ts
Inc. Ang.	$10^{16}$	45	90	0	0.25	0.2	π/2	1.0



Figure 2. Longitudinal (left plots) and vertical (right plots) components of FD and DWN synthetics for models A (first row) and B (second row) respectively. Each sub-plot label corresponds to the receiver number depicted in Figure 1. DWN synthetics are depicted with the red line, FD synthetics with PML thickness 30 grid spacings with the green line, with PML thickness 10 grid spacings with the blue line and with PML thickness 30 grid spacings for the enlarged model with black line.

The test was implemented by using eight profiles of ten receivers each, which correspond in eight incidence angles  $(0^0 \text{ to } 70^0)$  with a step of  $10^0$  as shown in Figure 1a. The deepest receiver had depth 7500 m and the shallowest one 2064 m. The longest receiver distance from the Y axis PML border was 4500 m and the closest one 150 m. For the enlarged model the longest distance is 8670 m and the closest one 4320 m. The source was placed at a selected point with Cartesian coordinates (990, 4500, 7500) m and at the point (5160, 4500, 7500) m for the enlarged model. In Figure 1a the configuration of source and receivers positions and in Figure 1b the configuration for the enlarged model, are shown.

For both models A and B, FD calculations were performed for two different PML thicknesses, 10 and 30 grid steps. For the enlarged model FD calculations were performed only for the 30 grid step PML thickness. The longitudinal and vertical components of the waveforms calculated for both models and PML thicknesses for the highlighted receivers of Figures 1 are shown in Figures 2-3 together with the DWN synthetics. In all Figures DWN synthetics are depicted with a red line, FD synthetics with PML thickness 30 grid spacings with a green line, FD synthetics with PML thickness 10 grid spacings with a blue line and FD synthetics for the enlarged model with a black line. For receiver 74 there was not available DWN synthetics because of an intrinsic limitation of the method which does not allow computing solutions for receivers that are at the same grid lines (horizontal or vertical) with the source.

## **COMPARISON WITH NON - REFLECTING BOUNDARIES**

The efficiency of PML comparing to other commonly used non-reflecting boundaries was also tested for the previously described model. Four different non-reflecting boundaries were used in this comparisons, [Clayton and Engquist, 1977], [Higdon, 1991] and [Peng & Toksöz, 1994;1995]. The computations were performed with the same source-receiver geometry and source parameters described in model A. The FD calculations were performed again for 10 and 30 grid steps PML thicknesses. The results for the same receivers as in Figures 2 are plotted in Figure 3. The synthetics calculated with the use of PML are shown in Figure 3 with the green, blue and black lines (30, 10 and enlarged model with 30 grid spacings thicknesses, respectively). Synthetics calculated with the use of [Clayton and Engquist, 1977] NRB are shown with the magenta line, [Higdon, 1991] with the green line and [Peng & Toksöz, 1994;1995] with the light blue and deep red lines respectively.



Figure 3. Longitudinal (left plot) and vertical (right plot) components of FD and NRB synthetics for model A. Each sub-plot label corresponds to the receiver number depicted in Figure 1. Synthetics calculated with the use of PML are shown with the green, blue and black lines (30, 10 and enlarged model with 30 grid spacings thicknesses, respectively). Synthetics calculated with the use of [Clayton and Engquist, 1977] NRB are shown with the magenta line, [Higdon, 1991] with the green line and [Peng & Toksöz, 1994;1995] with the light blue and deep red lines respectively.

For quantifying the results from these comparisons a modified Variance Reduction (mVR) coefficient (18) for each receiver and for all profiles shown in Figure 1a, was calculated using the DWN synthetics as the theoretical ones.

$$mVR = \frac{\int (Theoretical - Calculated)^2}{\int (Theoretical)^2} \times 100\%$$
(18)

The mVRs of all receivers, for each profile were averaged and plotted against the incidence angle. The results are shown in Figure 4, for the horizontal (left plot) and the vertical component (right plot) of the impinging wave.

From Figure 4 it is clear that PMLs (green and blue lines) are more efficient that the other non-reflecting boundaries. Moreover higher PML thicknesses result in higher PML efficiency. In general the behavior of the other non reflecting boundaries is expected; their performance is adequate for grazing incidence angles, but still lower than PMLs. In the critical angle a significant decrease of their efficiency is observed while for other angles their efficiency also continuous to decrease. Results from this comparison show the high efficiency of PMLs practically for all incidence angles and favorite their use against the specific non-reflecting boundaries used in this test.



Figure 4. Modified Variance Reduction coefficient plotted against angle of incidence for the longitudinal (left plot) and vertical (right plot) components of FD and NRB synthetics for model A. Each sub-plot label corresponds to the receiver number depicted in Figure 1. Synthetics calculated with the use of PML are shown with the green, blue and black lines (30, 10 and enlarged model with 30 grid spacings thicknesses, respectively). Synthetics calculated with the use of [Clayton and Engquist, 1977] NRB are shown with the green line and [Peng & Toksöz, 1994;1995] with the light blue and deep red lines respectively.

## PARAMETRIC STUDY OF THE ATTENUATING FUNCTION

The last test performed for the homogeneous models described previously was a parametric search for the attenuating function used in the PML in comparison with the PML thickness. For this test different values of the PML attenuation tuning variable,  $\tau$ , were used, spanning the [0.5-2] interval with a step of 0.5, for 10 and 30 grid steps PML thicknesses. The same computations as in the previous tests were repeated for model A and the modified Variance Reduction coefficient was calculated. The results are shown in Figures 5 and 6 for the horizontal and the vertical components correspondingly.

In both Figures 5 and 6 for the thinner PML zone the optimum value for the attenuating function used lies between 0.5-1.0 (light green-blue curves), while for the thicker PML zones higher values of attenuating function (1.0-1.5) must be used for optimum results (blue-red curves). From this parametric study, preliminary indications show that reflections due to the finite contrast in properties between successive grid nodes, in the thinner PML zones, could be eliminated with the usage of smaller attenuating values. On the other hand for thicker PML zones higher attenuating values could be used to enhance the overall PML performance.

## CONCLUSIONS

The implementation of a Non-Splitting formulation of Perfect Matching Layers technique was elaborated for the Cartesian coordinate system and implemented in a 3D-4<sup>th</sup> order staggered-grid velocity-stress finite-difference scheme. Numerical tests were performed for investigating its overall performance in homogeneous half space models with high and low Poisson ratios and the results were compared with the semi-analytical solution provided by the Discrete Wavenumber method. In these tests the Non-Splitting formulation did not exhibit any instability problems in any of the models used (high and low Poisson ratios). Their overall performance was the

one expected, more efficient in almost vertical incidence angles and less efficient at grazing angles. Moreover their efficiency for the critical angle of incidence depends on the PML thickness. The different PML thicknesses used led us to the trivial conclusion that thicker PML zones result in weaker artificial reflections. For computations with high accuracy demands a thicker PML zone is suggested (30 grid spacings) providing that the memory cost from the thickest PML zone is affordable. In terms of computational efficiency NPLM is equal to SPML, but in memory storage is slightly less efficient than SPML because more variables have to be kept in core memory.



Figure 5. Modified Variance Reduction coefficient plotted against angle of incidence for the longitudinal components of FD synthetics, for model A and different values of the PML attenuation tuning variable τ. The left figure corresponds to PML thickness 10 grid spacings and the right one to PML thickness 30 grid spacings.

The comparison of the Non-Splitting PML with other popular Non-Reflecting boundaries is clearly in favorite of PML. The statistical measure used for these comparisons, for any PML thickness and for all incidence angles exhibited systematically lower values. The superiority of PML is shown for both vertical and horizontal components. High values of the modified Variance Reduction for the vertical components are not only due to the poorer behavior of all non-reflecting boundaries in comparison with the longitudinal ones, but also to the very small amplitudes of the vertical components of the theoretical solution.



Figure 6. Modified Variance Reduction coefficient plotted against angle of incidence for the vertical components of FD synthetics, for model A and different values of the PML attenuation tuning variable τ. The left figure corresponds to PML thickness 10 grid spacings and the right one to PML thickness 30 grid spacings.

Results for both longitudinal and vertical components, for the thinner PML zone, show that optimum values for the attenuating functions are lower than the optimum values for the thicker one. This parametric study for the attenuating function used in the PML, may suggest that for thinner PML zones smaller values of attenuating function are needed to prevent artificial reflections. These reflections are probably due to the finite contrast in properties between successive grid nodes, suggesting that the discrete PML zone behaves as solid body. On the

contrary for thicker PML zones higher values of attenuating function should be used for reducing the reflections caused by the boundary used for the termination of the PML zone.

# APPENDIX: 3D – 2<sup>nd</sup> order Velocity-Stress staggered-grid finite - difference scheme for Non Splitting Formulation of Perfect Matching Layers.

$$\begin{split} T^{xy,m}_{l,\kappa,L+1/2} &= T^{xy,m-1}_{l,\kappa,L+1/2} + \mu_{l,\kappa,L+1/2} \Delta t \begin{bmatrix} \left| \frac{1}{h} \left( V^{m+1/2}_{l+1/2,\kappa,L+1/2} - V^{m+1/2}_{l-1/2,\kappa,L+1/2} \right) + P^{yx,m+1/2}_{l,\kappa,L+1/2} \right| + \\ \left| \frac{1}{h} \left( U^{m+1/2}_{l,\kappa+1/2,L+1/2} - U^{m+1/2}_{l,\kappa-1/2,L+1/2} \right) + P^{xy,m+1/2}_{l,\kappa,L+1/2} \right) \end{bmatrix} \\ T^{xz,m}_{l,\kappa+1/2,L} &= T^{xz,m-1}_{l,\kappa+1/2,L} + \mu_{l,\kappa+1/2,L} \Delta t \begin{bmatrix} \left| \frac{1}{h} \left( W^{m+1/2}_{l+1/2,\kappa+1/2,L} - W^{m+1/2}_{l-1/2,\kappa+1/2,L} \right) + P^{zx,m+1/2}_{l,\kappa+1/2,L} \right) + \\ \left| \frac{1}{h} \left( U^{m+1/2}_{l,\kappa+1/2,L+1/2} - U^{m+1/2}_{l,\kappa+1/2,L-1/2} \right) + P^{xz,m+1/2}_{l,\kappa+1/2,L} \right) \end{bmatrix} \\ T^{yz,m}_{l+1/2,\kappa,L} &= T^{yz,m-1}_{l+1/2,\kappa,L} + \mu_{l+1/2,\kappa,L} \Delta t \begin{bmatrix} \left| \frac{1}{h} \left( V^{m+1/2}_{l+1/2,\kappa,L+1/2} - V^{m+1/2}_{l+1/2,\kappa,L} \right) + P^{zy,m+1/2}_{l+1/2,\kappa,L} \right) \\ \left| \frac{1}{h} \left( W^{m+1/2}_{l+1/2,\kappa,L} - W^{m+1/2}_{l+1/2,\kappa,L-1/2} \right) + P^{zy,m+1/2}_{l+1/2,\kappa,L} \right) + \\ \left| \frac{1}{h} \left( W^{m+1/2}_{l+1/2,\kappa+1/2,L} - W^{m+1/2}_{l+1/2,\kappa-1/2,L} \right) + P^{zy,m+1/2}_{l+1/2,\kappa,L} \right) + \\ P^{ij,m+1/2}_{l,\kappa,L} &= T^{ij,m-1/2}_{l,\kappa,L} + \mu_{l+1/2,\kappa,L} \Delta t \begin{bmatrix} \left| \frac{1}{h} \left( W^{m+1/2}_{l+1/2,\kappa,L} - W^{m+1/2}_{l+1/2,\kappa-1/2,L} \right) + P^{zy,m+1/2}_{l+1/2,\kappa,L} \right) \\ \left| \frac{1}{h} \left( W^{m+1/2}_{l+1/2,\kappa+1/2,L} - W^{m+1/2}_{l+1/2,\kappa-1/2,L} \right) + P^{zy,m+1/2}_{l+1/2,\kappa,L} \right) \end{bmatrix} \end{bmatrix} \\ P^{ij,m+1/2}_{l,\kappa,L} &= e^{-\Omega'_{l,\kappa,L}\Delta t} P^{ij,m-1/2}_{l,\kappa,L} - \frac{1}{2} \Omega^{j}_{l,\kappa,L} \Delta t \left( e^{-\Omega'_{l,\kappa,L}\Delta t} \frac{\partial V^{m-1/2}_{l}}{\partial j} \right) \right|_{l,\kappa,L} + \frac{\partial V^{m+1/2}_{l}}{\partial j} \right|_{l,\kappa,L} ; K + \frac{\partial V^{m+1/2}_{l,\kappa,L}}{\partial j} \right|_{l,\kappa,L} ; K + \frac{\partial V^{m+1/2$$

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